MATHEMATICS MAGAZINE

Index to Volume 25 September 1951 - June 1952

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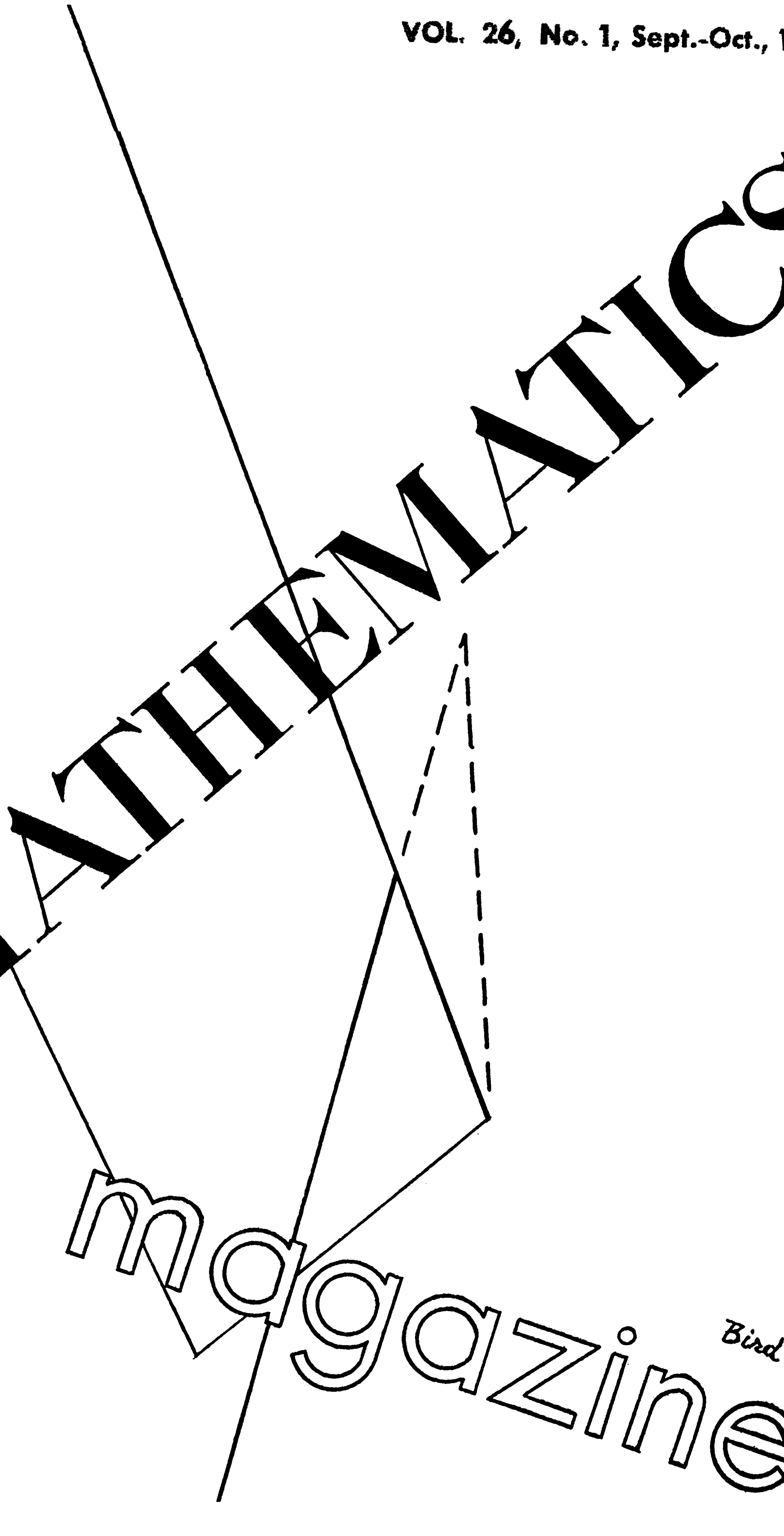
VOL. 26, No. 1, Sept.-Oct., 1952

MATHEMATICS magazine



MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S. T. Sanders.

Address editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to some member of the Executive Committee.

Manuscripts should be typed on 8½" x 11" paper, double-spaced with 1" margins. We prefer that, in technical papers, the usual introduction be preceded by a *foreword* which states in simple terms what the paper is about.

The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are 1 yr. $3.00; 2 yrs. $5.75; 3 yrs. $8.50; 4 yrs. $11.00; 5 yrs. $13.00. Sponsoring subscriptions are $10.00; single copies 65¢. Reprints: 25 free to authors; 100 @ $1.25 per page with a minimum of $6.00. All reprints are bound.

Subscriptions and related correspondence should be sent to Inez James, 14068 Van Nuys Blvd., Pacoima, Calif.

Advertisers should contact the managing editor.

Entered as second-class matter March 23, 1948 at the Post Office, Pacoima, California under act of congress of March 8, 1876.

MATHEMATICS MAGAZINE

Vol. 26, No. 1, Sept.-Oct., 1952

CONTENTS

CONSTRUCTION OF THE GREEN'S FUNCTION OF A LINEAR DIFFERENTIAL SYSTEM

Kenneth S. Miller

FOREWORD

The classical Green's function for a linear differential system is generally defined as a function of two variables with certain prescribed properties. In this paper we show how the concept of Green's function arises in a natural fashion from the familiar method of solving a non-homogeneous differential equation by the method of variation of parameters. The link between the two approaches is the one-sided Green's function – which has many useful properties in its own right.

1. *Introduction*

The Green's function is undoubtedly the most powerful tool that exists for investigating properties of ordinary linear differential systems. One of the elementary properties of this function is that it enables us to solve a non-homogeneous linear differential equation. That is, if we have a linear differential equation

$$Lu = p_0(x)u^{(n)} + p_1(x)u^{(n-1)} + \cdots + p_n(x)u$$

(where the coefficients are continuous in some closed finite interval $[a, b]$ and $p_0(x) > 0$ in $[a, b]$) and a set of boundary conditions

$$U_\alpha(u) = \sum_{i=1}^{n-1} A_{\alpha i} u^{(i-1)}(a) + \sum_{i=1}^{n-1} B_{\alpha i} u^{(i-1)}(b), \quad \alpha = 1, 2, \ldots, n$$

(such that the completely homogeneous system $Lu = 0$, $U_\alpha(u) = 0$, $\alpha = 1, 2, \ldots, n$ is incompatible) then $u(x)$ where

$$u(x) = \int_a^b G(x, \zeta) r(\zeta) d\zeta \tag{1}$$

is a solution of the non-homogeneous differential equation

$$Lu(x) = r(x)$$

which satisfies the boundary conditions

$$U_\alpha(u) = 0, \quad \alpha = 1, 2, \ldots, n.$$

The function $G(x, \zeta)$ which appears in Eq. (1) is called the *Green's function* of the system.

Another more elementary method for solving a linear differential equation is the classical method of variation of parameters. It would therefore appear that there should be an intimate relation between the two. This is indeed the case and we shall show how the concept of Green's function can be arrived at starting with the method of variation of parameters. In the course of our work certain interesting results will be obtained, namely, the idea of a "one-sided Green's function" which has many practical applications in itself, (e.g., in the theory of networks and feed-back circuits).

In most treatments, the Green's function is defined as a function of two variables which has a discontinuity in its $(n-1)st$ derivative with a finite jump of $1/p_0(\zeta)$ and formally satisfies $Lu = 0$ and $U_a(u) = 0$ except at the point $x = \zeta$. It is later shown that the function so defined has the desired properties, e.g., the result exhibited in Eq. (1). However, the development we are about to undertake appears to be a more natural argument in that it gives us a clear picture as to how the concept of Green's function arises logically from the familiar method of variation of parameters.

2. *The One-sided Green's Function*

Consider, for simplicity, a linear differential equation of the second order,

$$Lu = p_0(x)u'' + p_1(x)u' + p_2(x)u \tag{2}$$

where the coefficients p_0, p_1, p_2 are continuous in some closed finite interval $[a,b]$ and $p_0(x) > 0$ in this interval. We shall look for a solution $u(x)$ of the non-homogeneous equation

$$Lu = r(x)$$

(where $r(x)$ is continuous in $[a,b]$) which satisfies the homogeneous boundary conditions

$$u(a) = 0 \quad \text{and} \quad u'(a) = 0.$$

The method of variation of parameters tells us to assume a solution of the form

$$u(x) = v_1(x)\phi_1(x) + v_2(x)\phi_2(x)$$

where $\phi_1(x)$ and $\phi_2(x)$ are any two linear independent solutions of $Lu = 0$, and $v_1(x)$ and $v_2(x)$ are to be determined. Now,

$$u'(x) = v_1\phi_1' + v_2\phi_2'$$

and

$$u''(x) = v_1\phi_1'' + v_2\phi_2'' + r/p_0$$

where we have imposed the two conditions

$$\begin{aligned} v_1'\phi_1 + v_2'\phi_2 &= 0 \\ v_1'\phi_1' + v_2'\phi_2' &= r/p_0 \end{aligned} \tag{3}$$

on the functions v_1 and v_2. Forming Lu,

$$Lu = p_0(v_1\phi_1'' + v_2\phi_2'') + r + p_1(v_1\phi_1' + v_2\phi_2')$$

$$+ p_2(v_1\phi_1 + v_2\phi_2) = v_1(L\phi_1) + v_2(L\phi_2) + r = r$$

since ϕ_1 and ϕ_2 are solutions of $Lu = 0$.

$v_1(x)$ and $v_2(x)$ can be readily obtained from solving Eqs. (3). The determinant of these equations is

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix}$$

which is precisely the Wronskian, $W(x)$, of the linear differential operator of Eq. (2) and hence is never zero in $[a, b]$. Solving these simultaneous linear equations yields

$$v_1'(x) = \frac{-r(x)\phi_2(x)}{W(x)p_0(x)}, \quad v_2'(x) = \frac{r(x)\phi_1(x)}{W(x)p_0(x)}$$

and by integrating,

$$v_1(x) = -\int_a^x \frac{r(\zeta)}{p_0(\zeta)}\frac{\phi_2(\zeta)}{W(\zeta)}\,d\zeta + A$$

$$v_2(x) = \int_a^x \frac{r(\zeta)}{p_0(\zeta)}\frac{\phi_1(\zeta)}{W(\zeta)}\,d\zeta + B.$$

Hence the solution, $u(x)$ of $Lu = r(x)$ can be written:

$$u(x) = -\phi_1(x)\int_a^x \frac{r(\zeta)}{p_0(\zeta)}\frac{\phi_2(\zeta)}{W(\zeta)}\,d\zeta + \phi_2(x)\int_a^x \frac{r(\zeta)}{p_0(\zeta)}\frac{\phi_1(\zeta)}{W(\zeta)}\,d\zeta$$

$$+ A\phi_1(x) + B\phi_2(x)$$

which is the general solution of $Lu = r$. If now we combine the two integrals we have

$$u(x) = \int_a^x H(x,\zeta)r(\zeta)d\zeta + A\phi_1(x) + B\phi_2(x) \tag{4}$$

where

$$H(x,\zeta) = \frac{-1}{p_0(\zeta)W(\zeta)} \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1(\zeta) & \phi_2(\zeta) \end{vmatrix}. \tag{5}$$

Obviously, $u(x)$ defined by Eq. (4) has the property that $Lu(x) = r(x)$. We must now find the constants A and B such that $u(x)$ satisfies the prescribed boundary conditions, namely, $u(a) = 0$ and $u'(a) = 0$.

$$\begin{aligned} u(a) &= \int_a^a H(x,\zeta)r(\zeta)d\zeta + A\phi_1(a) + B\phi_2(a) \\ &= 0 + A\phi_1(a) + B\phi_2(a) = 0 \end{aligned} \tag{6}$$

and

$$u'(x) = H(x,x)r(x) + \int_a^x H_x(x,\zeta)r(\zeta)d\zeta + A\phi_1'(x) + B\phi_2'(x).$$

But from Eq. (5) we see that $H(x,x) = 0$ and hence

$$u'(a) = A\phi_1'(a) + B\phi_2'(a) = 0. \tag{7}$$

From Eqs. (6) and (7) we conclude $A = 0 = B$ since the determinant of the coefficients is precisely the Wronskian evaluation at $x = a$.

We can now state our conclusion in the form of a theorem.

THEOREM: Hypothesis. Let $Lu = p_0(x)u'' + p_1(x)u' + p_2(x)u$ be a linear differential equation where the coefficients p_0, p_1, p_2 are continuous functions in some closed finite interval $[a,b]$. Let $p_0 > 0$ in $[a,b]$. Let $\phi_1(x)$ and $\phi_2(x)$ be any two linear independent solutions of $Lu = 0$ with Wronskian $W(x)$. Define $H(x,\zeta)$ by Eq. (5).

Conclusion. If $r(x)$ is any continuous function in $[a,b]$ then the function $u(x)$,

$$u(x) = \int_a^x H(x,\zeta)r(\zeta)d\zeta$$

satisfies the non-homogeneous equation $Lu = r(x)$ and the boundary conditions

$$u(a) = 0, \quad u'(a) = 0.$$

This function $H(x,\zeta)$ is what we shall call a "one-sided Green's function". The definition arises because we are considering a *one-point* boundary value problem, that is, our boundary conditions $u(a)$ and $u'(a)$ depend on only one point of the interval. In general, we are interested in *two-point* boundary value problems, that is, where the initial conditions are given in the form

$$\begin{aligned} U_1(u) &= A_1 u(a) + A_2 u'(a) + A_3 u(b) + A_4 u'(b) \\ U_2(u) &= B_1 u(a) + B_2 u'(a) + B_3 u(b) + B_4 u'(b) \end{aligned} \tag{8}$$

and depend on both end points of the interval. Of course the *one-point* boundary value problems are included as special cases of these.

It is our purpose in the next section to show how the classical Green's function can be obtained from this one-sided Green's function. Before doing this we shall generalize the above theorem to differential equations of the nth order.

THEOREM: Hypothesis. Let $Lu = p_0(x)u^{(n)} + p_1(x)u^{(n-1)} + \cdots + p_n(x)u$ be a linear differential equation with the $p_i(x) \in C^0$ in some closed finite interval $[a, b]$. Let $p_0(x) > 0$ in $[a, b]$. Let $\{\phi_1(x), \phi_2(x), \ldots, \phi_n(x)\}$ be n linearly independent solutions of $Lu = 0$ with Wronskian $W(x)$,

$$W(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & \ldots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \ldots & \phi_n'(x) \\ \cdot & \cdot & \cdot & \cdot \\ \phi_1^{(n-1)}(x) & \phi_2^{(n-1)}(x) & \ldots & \phi_n^{(n-1)}(x) \end{vmatrix}.$$

Define

$$H(x,\zeta) = \frac{(-1)^{n-1}}{p_0(\zeta)W(\zeta)} \begin{vmatrix} \phi_1(x) & \phi_2(x) & \ldots & \phi_n(x) \\ \phi_1(\zeta) & \phi_2(\zeta) & \ldots & \phi_n(\zeta) \\ \phi_1'(\zeta) & \phi_2'(\zeta) & \ldots & \phi_n'(\zeta) \\ \cdot & \cdot & \cdot & \cdot \\ \phi_1^{(n-2)}(\zeta) & \phi_2^{(n-2)}(\zeta) & \ldots & \phi_n^{(n-2)}(\zeta) \end{vmatrix}.$$

Conclusion. If $r(x)$ is any continuous function in $[a, b]$, then the function $u(x)$,

$$u(x) = \int_a^x H(x,\zeta)r(\zeta)\,d\zeta$$

satisfies the non-homogeneous equation $Lu = r(x)$ and the boundary conditions $u^{(\beta)}(a) = 0$, $\beta = 0, 1, 2, \ldots, n-1$.

3. *The Green's Function*

We have seen in the first theorem that $H(x,\zeta)$ has the property that if

$$u(x) = \int_a^x H(x,\zeta)r(\zeta)d\zeta \tag{9}$$

then $Lu = r$ and $u(a) = 0 = u'(a)$. Suppose now our boundary conditions are given at the point b of the interval. That is, we wish to solve the equation $Lu = 0$ subject to the boundary conditions $u(b) = 0 = u'(b)$.

Clearly,

$$u(x) = -\int_x^b H(x,\zeta)r(\zeta)d\zeta \tag{10}$$

has these properties (as can be verified by a direct calculation). Now, let us combine these two results. Suppose we wish to solve $Lu = r$ subject to the *two-point* boundary conditions

$$u(a) + u(b) = 0$$

$$u'(a) + u'(b) = 0.$$

It is clear from Eqs. (9) and (10) that we want the Green's function to behave like $+\frac{1}{2}H(x,\zeta)$ in the neighborhood of $x = a$ and like $-\frac{1}{2}H(x,\zeta)$ in the neighborhood of $x = b$. Or to be more precise, define

$$g(x,\zeta) = \tfrac{1}{2}H(x,\zeta) \quad \text{if} \quad \zeta < x$$

$$g(x,\zeta) = -\tfrac{1}{2}H(x,\zeta) \quad \text{if} \quad \zeta > x.$$

We assert that if

$$u(x) = \int_a^b g(x,\zeta)r(\zeta)d\zeta \tag{11}$$

then $Lu = r$ and $u(a) + u(b) = 0$, $u'(a) + u'(b) = 0$.

To verify this assertion let us calculate $Lu = 0$ from Eq. (11). [This time we must be careful since $g(x,\zeta)$ and/or its derivatives may have discontinuities at $x = \zeta$.] We shall calculate $u'(x)$ and $u''(x)$.

$$u'(x) = \lim_{\substack{\zeta\to x\\ \zeta<x}} g(x,\zeta)r(\zeta) - \lim_{\substack{\zeta\to x\\ \zeta>x}} g(x,\zeta)r(\zeta) + \int_a^b \frac{\partial}{\partial x} g(x,\zeta)r(\zeta)d\zeta.$$

From the definition of $g(x,\zeta)$ we see that both the above limits are zero. Now,

$$u''(x) = \left[\lim_{\substack{\zeta\to x\\ \zeta<x}} \frac{\partial g(x,\zeta)}{\partial x} - \lim_{\substack{\zeta\to x\\ \zeta>x}} \frac{\partial g(x,\zeta)}{\partial x}\right] r(x) + \int_a^b \frac{\partial^2}{\partial x^2} g(x,\zeta)r(\zeta)d\zeta.$$

Since

$$\frac{\partial}{\partial x} g(x,\zeta) = \frac{-1}{2p_0(\zeta)} \begin{vmatrix} \phi_1'(x) & \phi_2'(x) \\ \phi_1(\zeta) & \phi_2(\zeta) \end{vmatrix} \qquad \zeta < x$$

we have

$$\lim_{\substack{\zeta\to x\\ \zeta<x}} \frac{\partial}{\partial x} g(x,\zeta) = \frac{+1}{2p_0(x)}.$$

Similarly,

$$\lim_{\substack{\zeta\to x\\ \zeta>x}} \frac{\partial}{\partial x} g(x,\zeta) = \frac{-1}{2p_0(x)}.$$

From which we conclude that

$$u''(x) = \frac{r(x)}{p_0(x)} + \int_a^b \frac{\partial^2}{\partial x^2} g(x,\zeta) r(\zeta) d\zeta$$

and hence indeed, $Lu = r(x)$. That this $u(x)$ satisfies the boundary conditions is also readily verified.

$$u(x) = \int_a^b g(x,\zeta) r(\zeta) d\zeta = \int_a^x \tfrac{1}{2} H(x,\zeta) r(\zeta) d\zeta - \int_x^b \tfrac{1}{2} H(x,\zeta) r(\zeta) d\zeta,$$

$$u(a) = \int_a^a - \int_a^b = -\int_a^b$$

and

$$u(b) = \int_a^b - \int_b^b = \int_a^b.$$

Hence $u(a) + u(b) = 0$. In a similar fashion we see that $u'(a) + u'(b) = 0$.

The function $g(x,\zeta)$ we have constructed is a Green's function in the classical sense. Of course, we have constructed it only for a *particular* set of two-point boundary conditions. Let us see how to form the Green's function $G(x,\zeta)$ for the same differential operator, namely, $Lu = p_0 u'' + p_1 u' + p_2 u$ and the most general two-point boundary conditions as given by Eq. (8). From what has been done above, we see that

$$G(x,\zeta) = g(x,\zeta) + \psi_1(\zeta)\phi_1(x) + \psi_2(\zeta)\phi_2(x) \tag{12}$$

has the property that if

$$u(x) = \int_a^b G(x,\zeta) r(\zeta) d\zeta,$$

then $Lu = r(x)$ whatever the functions $\psi_1(\zeta)$ and $\psi_2(\zeta)$ may be.

We shall now determine ψ_1 and ψ_2 such that $U_1(G)$ and $U_2(G)$ are zero. If we do this, then

$$U_i(u) = \int_a^b U_i\,[G(x,\zeta)]\,r(\zeta)d\zeta = 0, \quad i = 1,\ 2.$$

Hence this function $u(x)$ satisfies the boundary conditions of Eq. (8). From Eq. (12),

$$U_i(G) = U_i(g) + \psi_1(\zeta)U_i(\phi_1) + \psi_2(\zeta)U_i(\phi_2), \quad i = 1,\ 2.$$

If we equate these two equations to zero, we can solve for ψ_1 and ψ_2, namely:

$$\psi_1(\zeta) = \frac{\begin{vmatrix} -U_1(g) & U_1(\phi_2) \\ -U_2(g) & U_2(\phi_2) \end{vmatrix}}{\begin{vmatrix} U_1(\phi_1) & U_1(\phi_2) \\ U_2(\phi_1) & U_2(\phi_2) \end{vmatrix}} \qquad \psi_2(\zeta) = \frac{\begin{vmatrix} U_1(\phi_1) & -U_1(g) \\ U_2(\phi_1) & -U_2(g) \end{vmatrix}}{\begin{vmatrix} U_1(\phi_1) & U_1(\phi_2) \\ U_2(\phi_1) & U_2(\phi_2) \end{vmatrix}}.$$

[We place the restriction on the boundary conditions that

$$D(U) = \begin{vmatrix} U_1(\phi_1) & U_1(\phi_2) \\ U_2(\phi_1) & U_2(\phi_2) \end{vmatrix} \neq 0.$$

This is equivalent to saying that the system $Lu = 0$, $U_1(u) = 0$, $U_2(u) = 0$ is incompatible or that the only solution of $Lu = 0$, $U_1(u) = 0$, $U_2(u) = 0$ is $u \equiv 0$.]

A compact form of writing Eq. (12) is the following:

$$G(x,\zeta) = \frac{1}{D(U)} \begin{vmatrix} g(x,\zeta) & \phi_1(x) & \phi_2(x) \\ U_1(g) & U_1(\phi_1) & U_1(\phi_2) \\ U_2(g) & U_2(\phi_1) & U_2(\phi_2) \end{vmatrix}.$$

Of course, $G(x,\zeta)$ can be generalized to nth order linear differential systems.

New York University

SYSTEMS OF EQUATIONS, MATRICES AND DETERMINANTS

Olga Taussky and John Todd

INTRODUCTION

The solution of systems of simultaneous linear equations is a problem which has been before mathematicians for centuries. It is known, for instance, that the Babylonians studied systems of up to 10 equations. The solution of large systems arose in connection with the adjustment of astronomical and geodetical observations and it was in this connection that Gauss made notable contributions. More recently, considerable attention has been paid to large systems which arise by the approximation to differential and integral equations and in studies in economics.

The study of such systems is linked up with the theory of determinants. Fourier, in the original investigations concerning the coefficients in a trigonometrical series representing a periodic analytic function solved such systems (involving an arbitrary number of variables) without the use of determinants. The first systematic account of the theory of determinants is due to Cauchy. An axiomatic presentation is due to Weierstrass.

A most important aid to the study is the theory of matrices, which was introduced by Cayley in connection with the theory of transformations. Since then the theory of matrices has penetrated many branches of mathematics. Although matrices do not obey the commutative law of multiplication, they are of great service sometimes, in fact, for this very reason. They can be used to give representations of more abstract mathematical objects such as permutations, groups hypercomplex systems. Thus the theory has, therefore, many applications in such branches of mathematics as modern algebra, geometry, number theory, the theory of automorphic functions.

An obvious generalization of matrix theory is from the finite to the infinite case. This has been largely due to Hilbert and has proved of use for instance in the theory of integral equations, in the theory of divergent series and in modern theories of physics. It is important to note that the difference between the two theories is considerable, the finite case being algebraic, while the infinite is analytic.

The present article is not intended in any way to be complete; it presents only certain topics which, in our opinion, are interesting, important or useful. It has been divided into two chapters which are essentially self-contained: the first is theoretical and the second is practical.

CHAPTER I

I. 1. *DEFINITIONS, PRODUCTS AND SUMS, SPECIAL MATRICES, SIMILARITY TRANSFORMATIONS.*

A finite $n \times m$ matrix $A = (a_{ik})$, $i = 1, \ldots, n$; $k = 1, \ldots, m$, is a set of $n \cdot m$ real or complex numbers a_{ik}, usually arranged in a rectangular scheme

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & & a_{nm} \end{bmatrix}$$

If $n = m$ the matrix is called *square*, otherwise it is called *rectangular*. The elements a_{ik} for fixed i form the *i-th row*, for fixed k the *k-th column* of the matrix. If $n = m$ the elements a_{ii} form the *main diagonal* or *principal diagonal*.

The definition of matrices is only complete after the product and sum of two matrices has been defined. The product of two matrices is defined only in the case where the number of columns of the first matrix is equal to the number of rows of the second. If $A = (a_{ik})$ is an $n \times m$ matrix and $B = (b_{ik})$ is an $m \times p$ matrix then the product AB is the $n \times p$ matrix $C = (c_{ik})$ where

$$c_{ik} = a_{i1}b_{1k} + \cdots + a_{im}b_{mk}.$$

Thus the product of two $n \times n$ matrices is always defined. For example

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

The product of an $n \times m$ matrix A and a *column vector* i.e. an $n \times 1$ matrix, x, is defined and is written in the form Ax.

The product definition originates with the use of matrices in linear homogeneous transformations of variables. Let $x_1, \ldots, x_n$ be n variables and let the variables $y_1, \ldots, y_n$ be defined by the transformation

$$y_i = \sum_{k=1}^{n} b_{ik}x_k \quad \text{or} \quad y = Bx.$$

Let then $z_1, \ldots, z_n$ be another set of variables obtained from $y_1, \ldots, y_n$ by the transformation

$$z_i = \sum_{k=1}^{n} a_{ik} y_k \quad \text{or} \quad z = Ay.$$

It is easy to see that

$$z = ABx$$

in the sense of the matrix multiplication just defined.

A matrix which consists of zeros only is called a *zero matrix* and is frequently denoted by 0; a square matrix which has ones on the main diagonal and zeros elsewhere is called a *unit matrix* and is usually denoted by 1 or I, or by I_n if the matrix has n rows. It is easy to see that $I_n A = AI_n = A$ for all $n \times n$ matrices A. If all the elements off the principal diagonal are zero the matrix is called a principal diagonal matrix. If all the elements of a matrix below (above) the principal diagonal are zeros the matrix is called a right or upper (left or lower) *triangular matrix*.

Let A be an $n \times n$ matrix and let another $n \times n$ matrix B exist such that $AB = I$. It then follows that $BA = I$ and that B is unique. The matrix B is called the *inverse* of A and is denoted by A^{-1}. In general, however $AB \neq BA$. If $AB = BA$ the matrices A and B are said to *commute* or to be *commutative*. Another important point in which matrices differ from ordinary numbers is the fact that the product of two matrices can be the zero matrix without either of the factors being a zero matrix. It can even happen that the product of two matrices AB is a zero matrix, while BA is not a zero matrix, e.g., for

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

A matrix which has no inverse is called *singular*; it is *non-singular* if it has an inverse.

If a is a complex number and $A = (a_{ik})$ then aA is defined to be $(a\, a_{ik})$.

The sum of two matrices is defined only in the case where both have the same number of rows and columns. In this case, the sum of two $n \times m$ matrices $A = (a_{ik})$ and $B = (b_{ik})$ is defined as $C = (c_{ik})$ where

$$c_{ik} = a_{ik} + b_{ik}$$

If 0 is the $n \times m$ zero matrix then $0 + A = A + 0$ for all $n \times m$ matrices A. It is easily seen that both sum and product are associative operations and linked together by the distributive law whenever the product $A(B + C)$ is defined. The set of all $n \times n$ matrices forms a ring, a so-called *complete matrix algebra*. This algebra or hypercomplex system has n^2 base elements which can e.g. be chosen as the matrices with all elements but one equal to 0. The non-singular $n \times n$ matrices form a group under multiplication.

The matrix obtained by interchanging the rows and columns of a matrix A is called the *transpose* of A, and is denoted by A' or A^T. It is easily seen that $(AB)' = B'A'$. If the elements of A are complex numbers then $\bar{A}$ denotes the matrix obtained from A by replacing each element by its complex conjugate. The matrix $\bar{A}'$ is frequently denoted by A^*. A square matrix A for which $AA^* = A^*A$ is called *normal*. A special case of such matrices are the hermitian matrices for which $A = A^*$. A hermitian matrix with real elements is called *symmetric*. Another class of normal matrices is the *unitary* matrices, defined by $A^{-1} = A^*$. A real unitary matrix is called *orthogonal*.

An example of a normal matrix which is in general neither hermitian nor unitary is given by the *incidence* matrices, which occur in the study of finite projective planes. The elements of such a matrix are zero or one, with at least three ones in each row. Further, any two distinct rows (columns) have a one in common in exactly one column (row).

If in the equation $Ax = b$ (where x and b are column vectors and A is a square matrix) the vector b is replaced by Sb where S is a non-singular $n \times n$ matrix, then this new equation is equivalent to

$$S^{-1}AS(S^{-1}x) = b.$$

The matrix $S^{-1}AS$ is called similar to A or it is said to be obtained by a similarity *transformation* of A by S or that $S^{-1}AS$ is the transform of A by S. When a matrix A is transformed by another one certain quantities are left invariant, whatever S may be. One of them is the *trace* of A which is the sum of the elements on the principal diagonal. Another one is the determinant.

I. 2. *DETERMINANTS*

The *determinant* of A, also denoted by $|A|$ or $|a_{ik}|$ is the following number associated with the $n \times n$ matrix A:

$$\sum_{i_1, \dots, i_n} \pm a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

where $i_1, \dots, i_n$ runs through all permutations of the numbers $1, \dots, n$. The + sign is given to the terms where the permutation is even, the - sign when it is odd. The value of $|A|$ is rarely computed from this definition when $n > 3$. In order to compute it some of the following properties of $|A|$ are used:

(1) The value of $|A|$ is unaltered if a multiple of a row (or column) is added to another row (or column).

(2) If all the elements of a row (or column) are multiplied by a constant c then the determinant is multiplied by c.

(3) $|A| = |A'|$

(4) If two rows or columns are interchanged the determinant is multiplied by -1. This implies that the determinant of a matrix with two identical rows or columns vanishes; this fact could also be deduced from (1).

Conversely, it can be shown that any polynomial which is linear and homogeneous in the elements of each row of a matrix and which satisfies (4), and which reduces to 1 for unit matrices coincides with the determinant.

(5) $|A| = 0$ if and only if the rows are linearly dependent, i.e. if numbers $c_1, \ldots, c_n$ not all zero exist such that $\sum_{i=1}^{n} c_i a_{ik} = 0$, $k = 1, \ldots, n$.

(6) A matrix is non-singular if and only if $|A| \neq 0$.

(7) Denote by A_{ik} the determinant of the $n - 1 \times n - 1$ matrix obtained from A by omitting the i-th row and k-th column. Then

$$|A| = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} A_{ik}, \quad i = 1, \ldots, n,$$

$$0 = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} A_{jk} \quad (i \neq j).$$

The determinants A_{ik} are also called *minors* of A. They can be used to write down a formal inverse for a matrix $A = (a_{ik})$:

$$A^{-1} = \frac{1}{|A|} ((-1)^{i+k} A_{ik}).$$

However since the computation of determinants is rather cumbersome this expression is rarely used for numerical applications, and other methods for finding the inverse of matrices are employed. This will be discussed in Chapter II.

More general types of minors are obtained if $r (1 < r < n)$ rows and columns are omitted from A. If the rows and columns omitted have the same indices the minor is called a *principal minor*.

If A and B are two square matrices then

$$|AB| = |A| \; |B|$$

This implies the statement made earlier that the determinant of $S^{-1}AS$ is the same as that of A.

The concept of *rank* applies to matrices which are not necessarily square. Let A be any $n \times m$ matrix. Denote by S_q the $q \times q$ sub-matrix obtained by removing a certain $n - q$ rows and a certain $m - q$ columns from A. Let r be the largest value of q for which non-singular S_q exist. Then r is called the rank of A. E.g. if $a_{ik} \equiv 0$ then rank

$(a_{ik}) = 0$ while if $a_{ik} \equiv 1$, then rank $(a_{ik}) = 1$. If $m = n$ and A is non-singular it has rank n. It can be shown that the rank of A is the maximum number of linearly independent rows (or columns) in A.

I. 3. *CHARACTERISTIC ROOTS, CANONICAL FORMS, COMMUTATIVE MATRICES, QUADRATIC FORMS.*

The fact that the trace and determinant of A are unaltered under similarity transformation is only a consequence of the fact that the *eigenvalues* or *characteristic roots* of A are unaltered under such a transformation. These n numbers $\lambda_1, \ldots, \lambda_n$ are defined to be the values λ for which the matrix $(A - \lambda I_n)$ is singular, or alternatively as the roots of the determinantal equation

$$|A - \lambda I| = 0,$$

the so-called *characteristic equation* of A. Conversely, every algebraic equation $a_0 x^n + \cdots + a_{n-1}x + a_n = 0$ $(a_0 \neq 0)$ can be interpreted as the characteristic equation of an $n \times n$ matrix, e.g. of

$$\begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 0 & 1 \\ \frac{-a_n}{a_0} & \frac{-a_{n-1}}{a_0} & \cdot & \cdot & \frac{-a_2}{a_0} & \frac{-a_1}{a_0} \end{pmatrix}$$

The characteristic roots can also be defined as the numbers for which

$$Ax = \lambda x$$

for suitable vectors $x \neq 0$, the so-called *eigenvectors* or *characteristic vectors* (belonging to λ).

As an example consider the matrix $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. It has the characteristic roots $1 \pm 2i$. The vector $x = (x_1, x_2)$ which corresponds to $1 + 2i$ has $x_1 = 1$, $x_2 = i$; the vector which corresponds to $1 - 2i$ has $x_1 = -1$, $x_2 = i$.

If S is any non-singular $n \times n$ matrix then $S^{-1}(A - \lambda I)S = S^{-1}AS - \lambda I$; hence $|S^{-1}AS - \lambda I| = |A - \lambda I|$. This shows that a transformed matrix has the same characteristic equation as the original one. The trace of the matrix is the coefficient of $(-\lambda)^{n-1}$ in the characteristic equation, the determinant the constant term. The traces of matrices are used very intensively in the representation of groups by matrices.

The characteristic roots need not all be different. If a root is simple then the corresponding vector is uniquely determined apart from a common multiplier of its components. If a root is of multiplicity $r > 1$ then it can have more than one corresponding characteristic vector: e.g. the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has the double root $\lambda = 2$, while any vector (x_1, x_2) can be taken as corresponding characteristic vector; the matrix $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ also has the double root $\lambda = 2$, but only the vector $x_1 = 1$, $x_2 = 0$ corresponds to it.

There are other important quantities associated with a matrix which are not however, invariant under all similarity transformations. One of them is the *norm* of $A = \left(\sum_{i,k=1}^{n} |a_{ik}|^2 \right)^{\frac{1}{2}}$. This quantity is however invariant under all transformations by unitary matrices. The squared norm of A in the trace of $A\,A^*$.

The matrix $A\,A^*$ is hermitian. Such matrices have real characteristic roots; in particular real symmetric matrices have this property. Normal matrices do not, in general, have real characteristic roots; in fact a normal matrix with only real characteristic roots is hermitian. The matrix $A\,A^*$ has further all its characteristic roots non-negative. A hermitian matrix with only positive (non-negative) characteristic roots is called positive definite (positive semidefinite). Analogous definitions hold when the characteristic roots are negative (non positive). This notation originates from the link between hermitian matrices and forms described later.

It follows from the definition of the characteristic equation that the matrices A and A' have the same characteristic roots, while the characteristic roots of A^* and A are conjugate complex numbers. Although the characteristic roots of $A\,A^*$ are real and non-negative they are in general *not* the squared absolute values of the characteristic roots of A.

In general a product of two $n \times n$ matrices A, B does not have as characteristic roots the product of the characteristic roots of A and those of B for some ordering of the roots. It was, however, proved by Frobenius that commutative matrices have this property. Commutativity, however, is not a necessary condition. Pairs of matrices A,B which have the property that any polynomial $p(A,B)$ has as characteristic roots $p(\lambda_i, \mu_i)$ for some ordering of the roots λ_i of A and of the roots μ_i of B are called pairs of matrices with property P. They include all pairs of commutative matrices and depend on properties of the *commutator* $AB - BA$.

The characteristic roots of AB and BA always coincide. If, e.g. A is non-singular this follows easily from the fact that $BA = A^{-1}(AB)A$. It follows that the characteristic roots of a product $A_1 \ldots A_r$ are unaltered if the factors are permuted cyclically. This shows again that $S^{-1}AS$ and A have the same characteristic roots.

The similarity transform of a symmetric (hermitian) matrix is in general not symmetric (hermitian); however, the transform by means of an orthogonal (unitary) matrix is. It is further known that every hermitian matrix can be transformed by a unitary similarity into diagonal form i.e. into the form

$$\begin{bmatrix} \lambda_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \lambda_n \end{bmatrix}$$

with zeros off the principal diagonal. This is true whether the λ_i's are simple or not. The same is true for normal matrices (only the λ_i are then not all real), and this fact constitutes an alternative definition for matrices to be normal.

A general matrix cannot always be transformed by a similarity into diagonal form, e.g. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. If all the characteristic roots are simple then it is always possible. If there is a multiple root of multiplicity r (>1), but with less than r independent corresponding characteristic vectors then transformation to diagonal form is not possible by means of a similarity. If there are multiple roots then the matrix can in general only be transformed into triangular form, and this is always possible by a unitary transformation. A particular triangular form very frequently used is the Jordan normal form.

$$\begin{bmatrix} A_1 & & & & \\ & A_2 & & 0 & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & & \cdot \\ & & & & & A_r \end{bmatrix}$$

where A_i is of the form

$$\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_1 & & & 0 \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & & \cdot \\ & & & & & 1 \\ & & & & & \lambda \end{bmatrix}$$

A pair of matrices which has property P can be transformed to triangular form simultaneously by the same similarity transformation. A pair of matrices which can be transformed to diagonal form simultaneously by the same similarity transformation commutes.

Real symmetric matrices $A = (a_{ik})$ are closely connected with quadratic forms $\sum_{i,k=1}^{n} a_{ik}x_i x_k$ (or $x'Ax$). The fact that every quadratic form can be transformed into a sum of squares by a linear transformation of the variables corresponds to the fact that the matrix A can be transformed to main diagonal form by a transformation of the type $S'AS$. If the transforming matrix S is orthogonal so that $S' = S^{-1}$ then we obtain the *reduction to principal axes* for the quadratic form. Similar results hold for hermitian matrices and their associated form $\sum_{i,k=1}^{n} a_{ik}x_i \bar{x}_k$. *Skew symmetric* matrices are real matrices (a_{ik}) with $a_{ik} = -a_{ki}$. These matrices have as characteristic roots pairs of conjugate purely imaginary numbers or zero. If they are non-singular (this is only possible for even n) they can be transformed by orthogonal matrices to the form

$$\begin{bmatrix} 0 & \alpha_1 & & & & & \\ -\alpha_1 & 0 & & & & & \\ & & 0 & \alpha_2 & & & \\ & & -\alpha_2 & 0 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix}.$$

Any real matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix. Further, any matrix can be expressed in the form $H_1 + iH_2$ where H_1, H_2 are hermitian.

Symmetric non-singular matrices have symmetric inverses; their products, however, are in general not symmetric. In fact every matrix can be expressed as the product of two not necessarily real symmetric matrices. The product of two symmetric matrices will be symmetric matrices will be symmetric only when the matrices commute.

I. 4. *BOUNDS FOR CHARACTERISTIC ROOTS, FIELDS OF VALUES, POSITIVE MATRICES*

For many purposes it is not necessary to know the exact values of the characteristic roots, and certain estimates will suffice. Some of these estimates arise from considering the matrix $A A^*$ whose characteristic roots are non-negative. It can be shown that each $|\lambda_i|^2$ lies between the largest and the smallest characteristic roots of $A A^*$. Further, $\left\{ \sum_{i=1}^{n} |\lambda_i|^2 \right\}^{1/2}$ does not exceed the norm of A. Equality between these two sums holds if and only if A is normal. It is further known that $|\lambda_i| \leq \max_i r_i$, where r_i is the sum of the absolute values of the elements of the i-th row of A and an analogous theorem holds for the columns.

A great amount of information concerning the position of the characteristic roots is obtained from considering the *field of values* of the matrix. This is the set of complex numbers $\sum_{i,k=1}^{n} a_{ik}x_i\bar{x}_k$ where $x_1, \ldots, x_n$ is any set of numbers with $\sum_{i=1}^{n} x_i\bar{x}_i = 1$. The field of values is a closed convex and connected set which contains the characteristic roots inside or on the boundary. If A is normal then the set coincides with the smallest convex polygon which includes all the eigenvalues. If A is hermitian or real symmetric the field of values is a segment on the real axis and since A is then normal its end points are characteristic roots. This gives the well known fact that for a symmetric matrix the largest characteristic root is $\max \sum_{i,k=1}^{n} a_{ik}x_i\bar{x}_k$, where $\sum_{i=1}^{n} x_i\bar{x}_i = 1$, and the smallest characteristic root is $\min \sum_{i,k=1}^{n} a_{ik}x_i\bar{x}_k$.

If A is a matrix with positive elements only - a so-called *positive matrix* - then its characteristic root of maximum absolute value, the so-called *dominant root*, is real and positive and simple; its characteristic vector can be chosen to have all its components positive. Furthermore, only the dominant root can have such a vector.

If A is an arbitrary matrix (a_{ik}) and B a matrix with positive elements b_{ik} such that $|a_{ik}| \leq b_{ik}$ then any circle with center at 0 which includes all characteristic roots of B also includes the characteristic roots of A.

Of particular interest are the matrices which have all their characteristic roots inside the unit circle. These matrices are important in the study of the convergence of iterative processes.

Let us return to discussing bounds for roots of general matrices. An important class of bounds is obtained from the following theorem on determinants with *dominant principal diagonals*.

If $|a_{ii}| > \sum_{k\neq i} |a_{ik}|$, $i = 1, \ldots, n$, then $|A| \neq 0$. This theorem can be applied to the study of bounds for characteristic roots of matrices. It is applied to the characteristic determinant $|a_{ik} - \lambda\delta_{ik}|$, where $\delta_{ik} = 1$ for $i = k$ and 0 for $i \neq k$. It follows that this determinant can only vanish, i.e. λ can only be a characteristic root of (a_{ik}), if λ lies inside or on the boundary of one of the n circles with centers a_{ii} and radius $\sum_{k\neq i} |a_{ik}|$.

If the elements are all real and the principal diagonal elements positive then it can even be shown that the determinant of a matrix with dominant principal diagonal is positive, not merely different from zero. Moreover, the characteristic roots of such a matrix have positive real parts. If none of the non-diagonal elements of such a matrix is positive then the characteristic root with smallest real part is real. The latter results follow from the facts that a matrix with positive elements has a real dominant root, and that the given matrix can be expressed in the form $A = \alpha I - P$ where α is a positive number and P a positive matrix.

Matrices with non-negative principal diagonal and non-positive off diagonal elements have been studied extensively.

I. 5. *FUNCTIONS OF MATRICES, COMMUTATIVE MATRICES, LIMITS.*

The characteristic roots play an important role when functions of matrices are studied. If A is an $n \times n$ matrix then any polynomial in A can be expressed as a polynomial in A of degree less than n. The matrix A itself satisfies an equation of degree n, namely its own characteristic equation when the constant term c is replaced by cI_n and zero by the zero matrix. This constitutes the so-called *Cayley-Hamilton theorem*. The matrix can, however, satisfy an equation of degree less than n, and the equation of smallest possible degree is called the *minimum equation* of the matrix; e.g. the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has the characteristic equation $x^2 - 4x + 4 = 0$ and the minimum equation $x - 2 = 0$; the matrix $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ has characteristic and minimum equation both equal to $x^2 - 4x + 4 = 0$.

A rational function in A with non-singular denominator can again be expressed as a polynomial. Any polynomial in A commutes with A; a partial converse of this is the fact that any matrix which commutes with a matrix A which has only simple characteristic roots is a polynomial in A. Even if A and B commute it is not always possible to find a matrix C such that A and B are both polynomials in C.

The characteristic roots of a polynomial $p(A)$ in A are the numbers $p(\lambda_i)$ where λ_i are the characteristic roots of A. If A has all its characteristic roots inside the unit circle then $\lim_{n\to\infty} A^n = 0$, in the sense that if $A^n = (a_{ij}^{(n)})$ then $\lim_{n\to\infty} a_{ij}^{(n)} = 0$ for all i, j. Matrices with this property play an important role in the study of power series in A. It can, e.g. be shown that the inverse of $I - A$ can be expressed as the convergent power series $\sum_{i=1}^{\infty} A^i$ if and only if A has all its roots inside the unit circle. From a remark made earlier it follows that matrices of positive elements with row sums (or column sums) all less than 1 have all their characteristic roots inside the unit circle. These matrices are frequently used in probability theory.

We mentioned earlier that the set of all $n \times n$ matrices form a hypercomplex system with n^2 base elements. Conversely every hypercomplex system with n base elements can be "represented" by a subset of the set of $n \times n$ matrices. If the coefficients in the hypercomplex system are real numbers the coefficients in the representing matrix may be taken real too. By *representation* is meant a 1-1 correspondence which preserves addition and multiplication. Very well known examples of such representations are the 2×2 matrices $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ which represent the complex numbers $a + bi$ and the 4×4 matrices

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}$$

which represent the quaternions $a + bi + cj + dk$. There is a simple method of obtaining a representation by matrices for a hypercomplex system, the so-called *regular representation*. Let $e_1, \ldots, e_n$ be the base elements and $a_1 e_1 + \cdots + a_n e_n$ an arbitrary element of the system. The matrices (a_{ik}) defined by the relations $e_i(a_1 e_1 + \cdots + a_n e_n) = \Sigma a_{ik} e_k$ form the so-called *regular representation.*

(To be concluded in the Nov.-Dec. issue)

INFINITE SERIES AND TAYLOR AND FOURIER EXPANSIONS

(Concluded)

Robert C. James

Before using series in the solution of such problems, it is essential to know something about the nature of the sum function and to know under what conditions term by term differentiation and integration of an infinite series of variable terms is permissible. One of the most useful concepts to use in discussing these questions is uniform convergence. A series $u_1(x) + u_2(x) + \cdots$ converges to $S(x_0)$ for $x = x_0$ if for any positive number ϵ the error in approximating $S(x_0)$ by $\sum_{k=1}^{n} u_k(x_0)$ is less than ϵ for all n sufficiently large. If it is possible to find an N such that the error in approximating $S(x)$ by $\sum_{k=1}^{n} u_k(x)$ is small for all x in some region or interval I so long as $n \geq N$, the series is said to converge uniformly. More precisely:

A series $\sum_{k=1}^{\infty} u_k(x)$ is uniformly convergent on a set I if it is convergent for each x in I and, for any $\epsilon > 0$, there is an N (not dependent on x) such that $\left|s(x) - \sum_{k=1}^{n} u_k(x)\right| < \epsilon$ if $n \geq N$ and x is a point of I, where $S(x)$ is the sum of $\sum_{k=1}^{\infty} u_k(x)$.

Uniform convergence essentially requires that the remainder after n terms of the series can be made small for all x in the set I simply by taking n large enough. If there is a convergent series of non-negative constants M_n for which $|u_n(x)| \leq M_n$ for each x in I and each n, then the remainder of the variable series can be made small by taking n large enough to make the remainder of the series of constants small – and can thus be done independently of x. This test for uniform convergence is usually called the *Weierstrass M-test.*

If the terms of a series are all continuous at $x = x_0$, the sum of the first n terms is also continuous at x_0. But if the remainder is small for all x near x_0, then $S(x) - S(x_0)$ can not be much larger than the difference between the values of the sum of the first n terms at x and at x_0. This intuitive argument can be made rigorous to prove that *a series of continuous functions which converges uniformly has a continuous sum.*

Similarly, if the sum $S(x)$ is very nearly equal to $\sum_{k=1}^{n} u_k(x)$ at all points of an interval (a,b), then $\int_a^b S(x)dx$ must be very nearly equal to $\int_a^b [\sum_{k=1}^{n} u_k(x)]\,dx$, which is equal to $\sum_{k=1}^{n} \int_a^b u_k(x)dx$..This type of argument can be used to prove that a uniformly convergent series of continuous functions can be integrated term by term. It follows from this that a convergent series may be differentiated term by term if the derived series converges uniformly and each derivative is continuous.

3. *Power series.* A series of type $\sum_{k=1}^{\infty} c_k(x - a)^k$, where each c_k is a constant, is called a *power series.* By use of the ratio test, it can be shown that this series converges absolutely for $|x - a| < r$ if it converges for one value of x satisfying $|x - a| = r$. If it diverges for one value of x satisfying $|x - a| = r$, it diverges if $|x - a| > r$. Thus there is a constant R, called the *radius of convergence*, such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$. Use of the ratio test also shows that the series derived by differentiating each term of the series, or by integrating each term, has the same radius of convergence. It follows from the Weierstrass M-test (with $M_k = c_k r_0^k$) that if $r_0 < R$, then the series converges uniformly on the interval of all x satisfying $|x - a| \leq r_0$. Thus the terms of a power series may be differentiated or integrated within the interval of convergence, the result being the derivative or integral of the sum. As an example of this, consider the series $1 - x^2 + x^4 - x^6 + \cdots$. This is a geometric series which has the sum $1/(1 + x^2)$ if $|x| < 1$. By integrating the sum and the series between 0 and $1/\ 3$, one gets $\pi/6 = (1/\sqrt{3})(1 - 1/(3\cdot 3) + 1/(5\cdot 3^2) - 1/(7\cdot 3^3) + 1/(9\cdot 3^4) - \cdots)$.

4. *Taylor's series.* The principal uses of series of variable terms arise from the possibility of representing rather general functions as the sum of infinite series. Two types of series for which this is possible are power series and series of "orthogonal functions", both of which will be discussed briefly.

If one assumes that a given function $f(x)$ has a power series representation $c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots$ which converges to $f(x)$ for all x near $x = a$, the coefficients can be easily evaluated. Letting $x = a$ gives $c_0 = f(a)$. Differentiating $f(x)$ and the series and letting $x = a$ gives $c_1 = f'(a)$. Differentiating $f(x)$ and the series n times and letting $x = a$ gives $c_n = f^{(n)}(a)/n!$, where $f^{(n)}(a)$ is the nth derivative of $f(x)$ evaluated at $x = a$. With these values for the coefficients $\{c_n\}$, the power series is called a *Taylor's series* for the function $f(x)$. Unfortunately there are functions which are not the sum of their Taylor's series, even though the series are convergent. It is therefore essential to have some method for determining whether the sum of the series is the desired function, as well as whether the series converges. The fundamental test of this is whether the remainder

$f(x) - \sum_{k=0}^{n-1} c_k(x - a)^k = R_n(x)$ approaches zero as n becomes large.

If $f(x)$ has an nth derivative in the closed interval (a, x), then it can be shown that there is a number ξ between a and x for which $R_n(x) = [(x - a)^n/n!]f^{(n)}(\xi)$. There are other forms for $R_n(x)$, e.g. $R_n(x)$

$= [(x - a)(x - \xi)^{n-1}/(n - 1)!]f^{(n)}(\xi)$. In each case, ξ is a number between a and x, but possibly a different number in different forms of the remainder. If $n = 1$, the mean value theorem of differential calculus can be obtained by use of either of these forms of the remainder. In other words, when $n = 1$ one obtains $f(x)= f(a) + (x- a)f'(\xi)$ for some number ξ between a and x. In general, $R_n(x)$ is the error which results from approximating $f(x)$ by the polynomial of degree $n - 1$ which has the same value at $x = a$ and whose first $n - 1$ derivatives have the same values at $x = a$ as the corresponding derivatives of $f(x)$. The definition of convergence of an infinite series applied to this situation states that the series $\sum_{n=0}^{\infty} [(x - a)^n/n!]f^{(n)}(a)$ converges and has the sum $f(x)$ only for those x for which $\lim_{n\to\infty} R_n(x) = 0$.

For example, the Taylor's series for e^x with $a = 0$ is $1 + x + x^2/2! + x^3/3! + x^4/4! + \cdots$ with $R_n(x) = [x^n/n!]e^{\xi}$. The value of ξ in $R_n(x)$ is unknown, except for lying between 0 and x. If $x \geq 0$, $|R_n(x)| \leq |x^n/n!|$. Since $\lim_{n\to\infty} x^n/n! = 0$ for any value of x, it follows that $\lim_{n\to\infty} R_n(x) = 0$ and that the Taylor series for e^x converges for all x and the sum of the series is e^x.

Such an integral as $\int_0^1 e^{x^2}dx$ can be evaluated by means of the Taylor's series for e^x as $\int_0^1 (1 + x^2 + x^4/2! + x^6/3! + \cdots)dx = 1 + 1/3 + 1/(5\ 2!) + 1/(7\ 3!) + \cdots$. The error in approximating the integral by the first four terms of this series is $\int_0^1 R_4(x^2)dx = \int_0^1 [x^8/4!]e^{\xi}dx$, where $0 < \xi < x^2$ for each x. Since $x^2 \leq 1$ in the range of integration, the error is less than $e\int_0^1 (x^8/4!)dx = e/(9\cdot 4!)$.

The Taylor's series for $\ln x$ with $a = 1$ is $(x - 1) - (x - 1)^2/2 + (x - 1)^3/3 - (x - 1)^4/4 + \cdots$. In this case, $R_n(x) = [(x - 1)^n/n!] (-1)^{n+1}(n - 1)!/\xi^n$ for some ξ between 1 and x. If $x \geq 1$, $|R_n(x)| \leq (x - 1)^n/n$, which approaches zero as n becomes large if $x \leq 2$. If $x < 1$, it is better to use the second form of the remainder R_n given above. This gives $R_n(x) = (-1)^{n+1}[(x - 1)/\xi][(x - \xi)/\xi]^{n-1}$. For $0 < x < \xi < 1$, $|(x - \xi)/\xi| = (\xi - x)/\xi < (1 - x)$. Hence $\lim_{n\to\infty} R_n(x) = 0$ if $0 < x < 1$. It has thus been shown that the Taylor's series for $\ln x$ with $a = 1$ converges and has $\ln x$ as sum if $0 < x \leq 2$. If $x \leq 0$ or $x > 2$, the series diverges.

As another application of power series, consider finding the solution $y = y(x)$ of the differential equation $y'' + xy = e^x$ for which $y = 1$ and $y' = 1$ when $x = 0$. If one substitutes $\sum_{n=0}^{\infty} x^n/n!$ for e^x and $\sum_{n=0}^{\infty} a_n x^n$

for y, the equation becomes $2a_2 - 1 + \sum_{n=1}^{\infty} x^n[(n+2)(n+1)a_{n+2} + a_{n-1} - 1/n!] = 0$. This is equivalent to the equations $a_2 = 1/2$ and $a_{n+2} = -a_{n-1}/[(n+2)(n+1)] + 1/(n+2)!$, $n = 1, 2, \cdots$. Noting that $a_0 = 1$ and $a_1 = 1$ because of the conditions $y = 1$ and $y' = 1$ when $x = 0$, it is found that $y = 1 + x + \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{60}x^5 + \frac{1}{720}x^6 + \cdots$. It can be shown that for a large class of differential equations this method will yield a series with a positive radius of convergence which is a Taylor's series expansion of the solution of the differential equation.

5. *Expansions in series of orthogonal functions.* Let L be a set of functions defined on an interval (a,b) with the properties that the sum of two functions of L is a function of L and the product of a function of L and a real number is a function of L. Such a set L is an example of a linear vector space. If L is further restricted by the assumption that $\|f\| = [\int_a^b (f)^2 dt]^{1/2}$ exists for each function f belonging to L, then it can be shown that the quantity $(f,g) = \int_a^b (fg)dt$ exists for each pair f,g of members of L. If one calls $\|f\|$ the *length* of f and (f,g) the *inner product* of f and g, these concepts are very analogous to the same concepts for ordinary vectors. In fact, suppose that two functions u_1 and u_2 are orthogonal and let $f = a_1u_1 + a_2u_2$ and $g = b_1u_1 + b_2u_2$. Then $\|f\|^2 = \int_a^b (a_1u_1 + a_2u_2)^2 dt = a_1^2\|u_1\|^2 + a_2^2\|u_2\|^2$ and $(f,g) = \int_a^b (a_1u_1 + a_2u_2)(b_1u_1 + b_2u_2)dt = a_1b_1\|u_1\|^2 + a_2b_2\|u_2\|^2$, since $\int_a^b (uv)dt = 0$. If one normalizes u_1 and u_2 by making $\|u_1\| = \|u_2\| = 1$, then with this length and inner product the set of all finite linear combinations of u_1 and u_2 is equivalent to the ordinary two-dimensional vector space with u_1 and u_2 playing the role of unit perpendicular vectors.

If one has a sequence $u_1, u_2, u_3, \cdots$ of mutually orthogonal functions of unit "length" (called an *orthonormal sequence*), it is therefore natural to think of these as mutually perpendicular unit vectors and to investigate the possibility of expressing a function as an infinite series $\sum_{n=1}^{\infty} a_nu_n$, the coefficients $\{a_n\}$ being thought of as generalized components of the "vector" f. If u_1, u_2, u_3 were three mutually perpendicular unit vectors and $w = a_1u_1 + a_2u_2 + a_3u_3$, then, for $n = 1, 2$, or 3, the length of $(w - ku_n)$ is least for $k = a_n$. Likewise, the length of $w - (k_1u_1 + k_2u_2)$ is least for $k_1 = a_1$ and $k_2 = a_2$. By analogy, if a function f is representable by an infinite series $\sum_{n=1}^{\infty} a_nu_n$, one might expect the coefficients $\{a_n\}$ to be the numbers minimizing

$$\|f - \sum_{k=1}^{n} a_k u_k\|^2 = \int_a^b (f - \sum_{k=1}^{n} a_k u_k)^2 dt$$

$$= \int_a^b (f)^2 dt - 2\sum_{k=1}^{n} a_k \int_a^b (fu_k)dt + \sum_{k=1}^{n} a_k^{\,2}.$$

This is equivalent to minimizing $-2a_k\int_a^b (fu_k)dt + a_k^{\,2}$ for each k, which gives $a_k = \int_a^b (fu_k)dt$ for each k. This evaluation of the coefficients is also obtained if it is assumed that $f = \sum_{k=1}^{\infty} a_k u_k$ and that the convergence is such that one can integrate the series $fu_n = \sum_{k=1}^{\infty} (a_k u_k u_n)$ term by term. Since $\int_a^b u_k u_n = 0$ if $k \neq n$, and $\int_a^b (u_n)^2 dt = 1$, integration gives $\int_a^b (fu_n)dt = a_n$. With these values for the coefficients $\{a_n\}$, it follows that $\|f - \sum_{k=1}^{n} a_k u_k\|^2 = \int_a^b (f)^2 dt - \sum_{k=1}^{n} a_k^{\,2}$ for each n and that $\sum_{k=1}^{n} a_k^{\,2} \leq \int_a^b (f)^2 dt$ for each n (*Bessel's inequality*). Hence $\sum_{k=1}^{\infty} a_k^{\,2}$ is convergent and $\sum_{k=1}^{\infty} a_k^{\,2} \leq \int_a^b (f)^2 dt$. If $\sum_{k=1}^{\infty} a_k^{\,2} = \int_a^b (f)^2 dt$, then $\lim_{n\to\infty} \int_a^b (f - \sum_{k=1}^{n} a_k u_k)^2 dt = 0$ and $\sum_{k=1}^{n} a_k u_k$ is said to *converge in the mean* (of order two) to $f(x)$. If $\sum_{k=1}^{\infty} a_k^{\,2} = \int_a^b (f)^2 dt$ for each continuous function f when $a_k = \int_a^b (fu_k)dt$, the sequence $u_1, u_2, \cdots$ is said to be *complete* (this equality is then also satisfied for any Lebesgue measurable function whose square is Lebesgue integrable). It also follows that if $u_1, u_2, \cdots$ is a complete orthonormal sequence and $a_n = \int_a^b (fu_n)dt = 0$ for each n, then $\int_a^b (f)^2 dt = 0$. If f is continuous, this implies $f \equiv 0$. In general, it means that $f = 0$ at all except a set of points of measure zero. Conversely, it can be shown that u_1, $u_2, \cdots$ is complete if $\int_a^b (fu_n)dt = 0$ for each n implies $f = 0$ at all except a set of points of measure zero – whenever f is a Lebesgue measurable function whose square is Lebesgue integrable. In other words, the sequence is complete unless there is a function which is orthogonal to each member of the sequence.

6. *Fourier series.* The functions 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos 3x$, $\cdots$ are an orthogonal sequence on any interval $(a, a + 2\pi)$. In other words, $\int_a^{a+2\pi} \cos mx \sin nx\, dx = 0$ for all integers m and n, and $\int_a^{a+2\pi} \cos mx \cos nx\, dx = \int_a^{a+2\pi} \sin mx \sin nx\, dx = 0$ if $m \neq n$. This

sequence can also be shown to be complete, so that for any square integrable function $f(x)$ there is an infinite series $a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ which *converges in the mean* to $f(x)$ on the interval $(a, a + 2\pi)$. Since $\int_a^{a+2\pi} \cos^2 nx\, dx = \int_a^{a+2\pi} \sin^2 nx\, dx = \pi$, the sequence $1/\sqrt{2\pi}$, $(\cos x)/\sqrt{\pi}$, $(\sin x)/\sqrt{\pi}$, $(\cos 2x)/\sqrt{\pi}$, $\cdots$ is an orthonormal sequence and the coefficients in the expansion of $f(x)$ can be evaluated as discussed above. This gives:

$$a_n = \frac{1}{\pi} \int_a^{a+2\pi} f(x) \cos nx\, dx, \text{ if } n \geq 0;$$

$$b_n = \int_a^{a+2\pi} f(x) \sin nx\, dx, \text{ if } n \geq 1.$$

These formulas can also be obtained by multiplying $f(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ by $\cos nx$ or $\sin nx$ and integrating term by term over the interval $(a, a + 2\pi)$. A series $a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is called a *Fourier series* if there is a function $f(x)$ for which the coefficients $\{a_n\}$ and $\{b_n\}$ are given by the above formulas. If Lebesgue integration is used, this can be shown to be the case if $\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$ is convergent. In this case, it is also true that:

$$\frac{1}{2}\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \int_a^{a+2\pi} (f)^2 dt.$$

For an example of a Fourier series expansion, let $f(x)$ be defined on the interval $(-\pi, \pi)$ by $f(x) = 0$ when $-\pi < x \leq 0$ and by $f(x) = x$ when $0 \leq x < \pi$. Then:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} x\, dx = \pi/2,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{\pi} x(\cos nx) dx = (\cos n\pi - 1)/(\pi n^2),$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_0^{\pi} x(\sin nx) dx = -(\cos n\pi)/n.$$

The Fourier series for $f(x)$ is therefore

$$\frac{1}{4}\pi + [-\frac{2}{\pi} \cos x + \sin x] - \frac{1}{2} \sin 2x$$

$$+ [-\frac{2}{9\pi} \cos 3x + \frac{1}{3} \sin 3x] - \frac{1}{4} \sin 4x + \cdots.$$

This series converges to $f(x)$ for each x satisfying $-\pi < x < \pi$, and converges to $\pi/2$ if $x = -\pi$ or $x = +\pi$. In the latter case, the series becomes $\pi^2/8 = 1 + 1/3^2 + 1/5^2 + 1/7^2 + \cdots$.

Unfortunately, the Fourier series of a function may converge *in the mean* to the function, even though for particular values of x the series may not have a sum equal to the value of the function at x. Point-wise convergence must be established by essentially different methods than convergence in the mean. For this purpose, the concepts of right-hand and left-hand derivatives and of bounded variation are very useful. The *right-hand* and *left-hand derivatives* of $f(x)$ at $x = x_0$ are defined, respectively, as $\lim_{h\to 0} [f(x_0 + h) - f(x_0 + 0)]/h$ and $\lim_{h\to 0} [f(x_0 - h) - f(x_0 - 0) /h$, if these exist under the restriction that h is positive. The symbols $f(x_0 + 0)$ and $f(x_0 - 0)$ are used for $\lim_{h\to 0} f(x_0 + h)$ and $\lim_{h\to 0} f(x_0 - h)$, if these exist under the restriction that h is positive. A function $f(x)$ is of *bounded variation* in an interval (a, b) if there is a number M such that for all increasing finite sequences of numbers $x_1 = a, x_2, x_3, \cdots, x_n = b$ it is true that $\sum_{k=1}^{n-1} |f(x_{k+1}) - f(x_k)| < M$.

With the above values of the coefficients, the Fourier series for $f(x)$ can be written as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \frac{1}{\pi}\sum_{n=1}^{\infty} [\cos nx \int_a^{a+2\pi} f(t)\cos nt\, dt$$

$$+ \sin nx \int_a^{a+2\pi} f(t)\sin nt\, dt].$$

By use of trigonometric identities, the sum of the first $n + 1$ terms of this series can be written as:

$$S_n(x) = \frac{1}{\pi}\int_a^{a+2\pi} f(t)\{½ + \sum_{k=1}^{n} \cos[k(t - x)]\}dt$$

$$= \frac{1}{\pi}\int_a^{a+2\pi} f(t)\,\frac{\sin[(n + ½)(t - x)]}{2\sin[½(t - x)]}\,dt.$$

Let $f(x)$ be defined, or redefined, outside the interval $(a, a + 2\pi)$ so that $f(x + 2\pi) = f(x)$ for all x. Then if $\int_a^{a+2\pi} f(t)dt$ exists and, if this is an improper integral, $\int_a^{a+2\pi} |f(t)|dt$ exists, then $\lim_{n\to\infty} S_n(x_0)$ can be shown to depend only on the nature of $f(x)$ for x near x_0. Furthermore, it can be shown that $\lim_{n\to\infty} S_n(x_0)$ (the sum of the Fourier series) exists and equals $½[f(x_0 + 0) + f(x_0 - 0)]$ if any one of the following conditions is also satisfied:

i) x_0 is an interior point of some interval in which $f(x)$ is of bounded variation.

ii) The right-hand and left-hand derivatives of $f(x)$ exist at x_0.

iii) $\int_0^k |[f(x_0 + t) - f(x_0 + 0) + f(x_0 - t)] - f(x_0 - 0)]/t|dt$ exists for some positive number h.

A convergent Fourier series does not necessarily converge uniformly. However, if $f(x)$ is continuous and of bounded variation in an interval (a, b) and $0 < \epsilon < b - a$, then the Fourier series for $f(x)$ will converge uniformly to the sum $f(x)$ in the interval $(a + \epsilon,\ b - \epsilon)$.

Trigonometric expansions of a function on an arbitrary interval can be obtained from a Fourier series by a change in variable. Thus for an interval $(a, a + p)$:

$$f(x) = \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} [a_k \cos(2\pi kx/p) + b_k \sin(2\pi kx/p)],$$

where $a_n = \dfrac{2}{p}\displaystyle\int_a^{a+p} f(t)\cos(2n\pi t/p)dt$ and $b_n = \dfrac{2}{p}\displaystyle\int_a^{a+p} f(t)\sin(2n\pi t/p)dt.$

Let $f(x)$ be defined on an interval $(0,L)$ and be extended to the interval $(-L,0)$ by $f(x) = f(-x)$, $f(x)$ then being an *even function* on $(-L,L)$. Letting $a = -L$ and $p = 2L$ in the above then gives the *Fourier cosine series:*

$$f(x) = \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{L}, \text{ where } a_k = \frac{2}{L}\int_0^r f(t)\cos\frac{k\pi t}{L}\, dt.$$

Similarly, if $f(x)$ is extended by $f(-x) = -f(x)$ so as to be an *odd function*, one gets the *Fourier sine series:*

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{L}, \text{ where } b_k = \frac{2}{L}\int_0^r f(t)\sin\frac{k\pi t}{L}\, dt.$$

7. *Some applications to the solution of boundary value problems.* Consider the problem of finding the vertical displacement $y(x,t)$ of an elastic string of length L stretched between the points $(0,0)$ and $(L,0)$ if at time $t = 0$ it is displaced into a position $y = f(x)$ and released. Under certain restrictions and for a suitable constant a, $y(x,t)$ will satisfy the partial differential equation $\partial^2 y/\partial^2 t^2 = a^2(\partial^2 y/\partial x^2)$. The following boundary conditions must also be satisfied: $y(0,t) = y(L,t) = 0$ for all $t \geq 0$, $y(x,0) = f(x)$ for $0 \leq x \leq L$, and $\delta y(x,0)/\delta t = 0$ for $0 \leq x \leq L$. The function $\sin(n\pi x/L)\cos(n\pi at/L)$ satisfies the partial differential equation. If n is an integer, it satisfies all of the boundary conditions except $y(x,0) = f(x)$. The same is also true for any finite sum $\sum_{k=1}^{n} c_k \sin(k\pi x/L)\cos(k\pi at/L)$. It might then seem reasonable to try to determine constants $\{c_k\}$ for

which $y = \sum_{k=1}^{\infty} c_k \sin(k\pi x/L)\cos(k\pi at/L)$ is the desired solution. The remaining boundary condition then becomes $f(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x/L)$ for $0 \leq x \leq L$. If $c_k = \frac{2}{L}\int_0^r f(t)\sin(k\pi t/L)dt$, this is the Fourier sine series for $f(x)$ in the interval $(0,L)$. With suitable restrictions on $f(x)$, it can be shown that, with these values for the coefficients $\{c_k\}$, $y = \sum_{k=1}^{\infty} c_k \sin(k\pi x/L)\cos(k\pi at/L)$ is the desired solution.

Let us next consider the problem of finding the steady-state temperature distribution of a circular plate whose circumference is kept at a prescribed temperature and whose faces are insulated. If polar coordinates are used, Laplace's equation (which the temperature $U(r,\theta)$ must satisfy) has the form:

$$\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial^2 y}{\partial \theta^2} + \frac{1}{r}\frac{\partial y}{\partial r} = 0.$$

If one substitutes $U = f(r)g(\theta)$, it is found by separating variables in the resulting equation that $(c_1 r^k + c_2 r^{-k})(c_3 \cos k\theta + c_4 \sin k\theta)$ is a solution of the equation for arbitrary values of c_1, c_2, c_3, c_4, and k. If one sets $c_2 = 0$ so the solution will be continuous inside the circle, this can be written in the form $c_k r^k \cos k\theta + d_k r^k \sin k\theta$. For suitable choice of the coefficients, the series $U(r,\theta) = \frac{1}{2} c_0 + \sum_{k=0}^{\infty} (c_k r^k \cos k\theta + d_k r^k \sin k\theta)$ will be the desired solution. If the circle is of radius ρ and the temperature on the boundary of the circle is given by $U(\rho,\theta) = f(\theta)$, we must have $f(\theta) = \frac{1}{2} c_0 + \sum_{k=1}^{\infty} (c_k \rho^k \cos k\theta + d_k \rho^k \sin k\theta)$. If this is to be the Fourier series for $f(\theta)$, then $c_k = \frac{1}{\pi\rho^k}\int_0^{2\pi} f(\theta)\cos k\theta \, d\theta$ and $d_k = \frac{1}{\pi\rho^k}\int_0^{2\pi} f(\theta)\sin(k\theta)d\theta$.

8. *Legendre polynomials.* It seems appropriate to discuss at least one of several sequences of orthogonal functions which are of great value in applied mathematics. The Legendre polynomials are a sequence of polynomials $P_0 = 1$, $P_1 = x$, $P_2 = \frac{1}{2}(3x^2 - 1)$, $\cdots$ such that $P_n(x)$ is of degree n and $\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$ if $m \neq n$. This determines the polynomials except for multiplicative constants. More explicitly,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n(z^2-1)^n}{dz^n} = \sum_{k=0}^{m} (-1)^k \frac{(2n-2k)!}{2^n \cdot k!\,(n-k)!\,(n-2k)!} z^{n-2k},$$

where $m = n/2$ or $(n-1)/2$, whichever is an integer.

The Legendre polynomials form a complete orthogonal sequence on the interval (-1,1). The expansion of a function in a series of Legendre, polynomials therefore *converges in the mean* on the interval (-1,1) to the function and has the form $f(x) = \sum_{n=0}^{\infty} A_n P_n(x)$, where $A \int_{-1}^{1} [P_n(x)]^2 dx = \int_{-1}^{1} f(x)P_n(x)dx$. It can be shown that $\int_{-1}^{1} [P_n(x)]^2 dx = 1/(n + \frac{1}{2})$, so this reduces to

$$A_n = (n + \tfrac{1}{2}) \int_{-1}^{1} f(x)P_n(x)dx.$$

Sufficient conditions for point-wise convergence are more restrictive than for convergence in the mean, but are very similar to those given for Fourier series. In fact, if $\int_{-1}^{1} (1 - t^2)^{-1/4} f(t)dt$ and $\int_{-1}^{1} |(1 - t^2)^{-1/4} f(t)|dt$ exist and $-1 < x_0 < 1$, then $\sum_{k=0}^{\infty} A_n P_n(x)$ will converge and have the sum $\frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)]$ if any one of the conditions *i*, *ii*, or *iii* given for the convergence of a Fourier series is satisfied.

Any solution of Laplace's equation is called a *harmonic function*. Legendre's polynomials are cllled *zonal harmonics* because of their use in such problems as the following, where U is constant on boundaries of zones of a spherical surface: Let U be the steady state temperature of a sphere whose surface is kept at a prescribed temperature, or let U be the electric potential in a sphere whose surface is kept at a fixed potential and whose interior is free of charges. If U is assumed to be symmetrical about the z-axis and spherical coordinates (r,θ,ϕ) are used with the origin at the center of the sphere and ϕ the angle between the z-axis and the radius vector, then U will be independent of θ and Laplace's equation (which U must satisfy) reduces to:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left\{ r^2 \frac{\partial u}{\partial r} \right\} + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left\{ \sin \phi \frac{\partial u}{\partial \phi} \right\} = 0.$$

It can be verified that $r^n P_n(\cos \phi)$ is a solution of this equation for $n = 0, 1, 2, \cdots$. If the radius of the sphere is ρ and $U = f(\phi)$ on the surface of the sphere, then U will be equal to $\sum_{n=0}^{\infty} A_n r^n P_n(\cos \phi)$ if the coefficients $\{A_n\}$ are determined so that $f(\phi) = \sum_{n=0}^{\infty} A_n \rho^n P_n(\cos \phi)$. Writing $f(\phi) = F(\cos \phi)$, it is seen that $A_n \rho^n = (n + \frac{1}{2}) \int_{-1}^{1} F(x)P_n(x)dx = (n + \frac{1}{2}) \int_{0}^{\pi} f(\phi)P_n(\cos \phi)\sin \phi \, d\phi$.

9. *Other convergence processes.* The following two infinite processes will be discussed briefly in an attempt to widen the reader's concept of the variety of such processes, other examples of which are improper integrals, double and multiple series, continued fractions,

infinite determinants and matrices, and asymptotic series.

An *infinite product* is an indicated product $(1+a_1)(1+a_2)(1+a_3)\cdots$ of an infinite number of factors. It is said to converge if $\lim_{n\to\infty}(1+a_1)(1+a_2)\cdots(1+a_n)$ exists and is not zero. If $a_n \neq -1$ for any n, then a sufficient condition for convergence is that $\sum_{n=1}^{\infty}|a_n|$ converge. If $C \leq a_n$ for each n, or $-1 \neq a_n \leq 0$ for each n, this is also a necessary condition for convergence. If $\sum_1^{\infty}|a_n|$ converges, the product is said to *converge absolutely*. A very useful and interesting infinite product is:

$$\sin x = x[1-(x/\pi)^2][1-(x/2\pi)^2][1-(x/3\pi)^2]\cdots.$$

There are various ways of assigning a "sum" to certain divergent series. If this is possible, the series is said to be *summable* relative to that particular method. A basic requirement of all such methods is that when applied to a convergent series they give the usual sum. The following is known as Cesaro's method of summation. Let $a_1 + a_2 + \cdots$ be an infinite series and $s_n = a_1 + a_2 + \cdots + a_n$. If $\lim_{n\to\infty}(s_1 + s_2 + \cdots + s_n)/n = S$ exists, then $\sum_1^{\infty} a_n$ is said to be *summable (C1)* and its sum (C1) is S. One of the most interesting applications of this method is to Fourier series. It can be shown that if $f(x)$ is periodic of period 2π and $\int_0^{2\pi} f(t)\,dt$ exists and, if this is an improper integral, $\int_0^{2\pi}|f(t)|\,dt$ exists, then the Fourier series for $f(x)$ is summable (C1) at all points x_0 for which $f(x_0+0)$ and $f(x_0-0)$ exist. Also, its sum (C1) is $\frac{1}{2}[f(x_0+0)+f(x_0-0)]$.

Haverford College

COLLEGIATE ARTICLES

Graduate training not required for reading

SPHERES ASSOCIATED WITH A TETRAHEDRON

Victor Thébault

1. Let $ABCD$ be a tetrahedron whose edges BC and DA, CA and DB, AB and DC have lengths a and a', b and b', c and c'. We extend the edges (DA, DB, DC), ..., from each vertex by k times their lengths so that the twelve points obtained (D'_a, D'_b, D'_c), (A'_b, A'_c, A'_d), ..., are such that $\vec{AD'_a} = k\,\vec{DA}$, $\vec{BD'_b} = k\,\vec{DB}$, $\vec{CD'_c} = k\,\vec{DC}$, ...

It is proposed to evaluate the sum of the powers of each of the points (D'_a, D'_b, D'_c), ..., with respect to the sphere described on the opposite edge as a diameter.

The sum of the powers $P(D'_a)$, $P(D'_b)$, $P(D'_c)$ of points D'_a, D'_b, D'_c with respect to the spheres described on the edges BC, CA, AB as diameters is equal to

$$\overline{D'_aA_1}^2 - \overline{BC}^2/4 + \overline{D'_bB_1}^2 - \overline{CA}^2/4 + \overline{D'_cC_1}^2 - \overline{AB}^2/4$$

where A_1, B_1, C_1 are the midpoints of BC, CA, AB.

If we apply Stewart's theorem to triangles $DA_1D'_a$, $DB_1D'_b$, $DC_1D'_c$ to evaluate $\overline{D'_aA_1}^2$, $\overline{D'_bB_1}^2$, $\overline{D'_cC_1}^2$, we obtain

$$P(D'_a) = (k+1)(b^2+c^2)/2 + k(k+1)a'^2 - k(b'^2+c'^2)/2 - a^2/2,$$

$$P(D'_b) = (k+1)(c^2+a^2)/2 + k(k+1)b'^2 - k(c'^2+a'^2)/2 - b^2/2,$$

$$P(D'_c) = (k+1)(a^2+b^2)/2 + k(k+1)c'^2 - k(a'^2+b'^2)/2 - c^2/2,$$

By suitable permutations we find analogous formulas for the powers of the nine other points (A'_b, A'_c, A'_d), ..., with respect to the spheres described on the opposite edges as diameters.

The sum of the powers of the twelve points in question is equal to

$$\Sigma\, P(D'_a) = (2k^2 + 2k + 1)\ \Sigma\,(a^2 + a'^2).$$

2. If we consider the twelve points (D''_a, D''_b, D''_c), ..., situated on the edges (DA, DB, DC), ..., such that

$$\vec{DD''_a} = k\,\vec{DA},\ \ldots,$$

we obtain in the same manner

$$P(D''_a) = k(b^2 + c^2)/2 + k(k-1)a'^2 - (k-1)(b'^2 + c'^2)/2 - a^2/2, \cdots$$

and

$$\Sigma\, P(D''_a) = (2k^2 - 2k + 1)\ \Sigma\ (a^2 + a'^2).$$

In both these cases the sums of the powers in question are positive, for

$$2k^2 + 2k + 1 = k^2 + (k+1)^2, \quad 2k^2 - 2k + 1 = k^2 + (k-1)^2.$$

3. Special cases:

1). $k = -1,\ \ \Sigma\, P(D'_a) = \Sigma\ (a^2 + a'^2).$

2). $k = 0,\ \ \Sigma P(D'_a) = \Sigma\, P(D''_a) = \Sigma\ (a^2 + a'^2).$

THEOREM. In a tetrahedron the sum of the powers of the vertices with respect to the spheres described on the edges of the opposite faces as diameters is equal to the sum of the square of the edges.

3). $k = 1,\ \Sigma\, P(D'_a)\ 5\ \Sigma\ (a^2 + a'^2),\ \Sigma\, P(D''_a) = \Sigma\ (a^2 + a'^2).$

An analogous question for the triangle has been discussed by C. A. Laisant, Nouvelle Correspondance Mathematique, Vol. 3, 1877, p. 394.

Translated from the French by W. E. Byrne.

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

A REFLECTIVE APPROACH IN THE TEACHING OF MATHEMATICS

Charles A. Johnson

The purpose of this article is to illustrate a reflective approach in the teaching of a topic in mathematics through the study of ideas in contrast or conflict.

A word of explanation of the pedagogical principles involved is in order. The technique is essentially one of posing a provocative question at the outset, allowing the class time to consider possible solutions, and, finally, adopting a conclusion. The choice of the question is governed by its potentialities for throwing light on conflicts in ideas which reveal inadequacies or disharmonies in the outlook of the student. The question may seem to the class to concern something that they already thoroughly understand and will often be such as to evince a definite "yes" or "no" response. Then the follow-up question comes: "How do you know?" It is essential to the method that questions come first and conclusions last. Between the question and the conclusion there must be ample time for the class to reflect upon alternative hypotheses which they have formulated in attempting to answer the question. Fundamentally, it is the application of the scientific method in the classroom consideration of problems. It is the antithesis of the "telling" method of teaching.

I shall illustrate the technique by outlining the procedure which might be followed in presenting Maclaurin's Series.

It is assumed that what follows has been immediately preceded by by a study of power series and their properties as well as the ideas of convergence and divergence of series.

We may begin by considering the following relations:

$$1/(1 - x) = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$(1 - 2)^{-1} = 1 + 2 + 4 + 8 + \cdots$$

$$(1 + x)^{½} = 1 + x/2 - x^2/2^2 3! + 3x^3/2^3 3! - \cdots$$

The first of these may be obtained by a simple process of division while the second and third may be presented as applications of the Binomial Theorem. This is the approach suggested in Middlemiss' "Differential and Integral Calculus".

Now the class is confronted with this question: Are these relations valid? Of course, the student may have a definitely apathetic attitude toward the question because he does not immediately appreciate the consequences of his answer. A student in one of my classes recently reacted submissively with the comment, "Math. is math." Many, in fact, will see no sense at all to the question. But if they are challenged to substantiate a "yes" or "no" response the discussion will be likely to proceed as follows:

"They were obtained by legitimate processes, so it seems to me that they should be valid. But the second one is obviously an absurdity."

"Yes, and if you put x equal to 4 in the first one, you get another absurd result."

"But if you put x equal to ½ in the first one you obtain a relation which we know to be true."

The class now appreciates the conflict because it realizes that the relations under consideration are both "true" and "false" depending on the choice of value for x. We proceed to a discussion of this question: For what values of x do these relations seem to be valid?

"The first relation seems to hold for $|x| < 1$."

What set of values does this represent?

"The interval of convergence of the power series."

Now we proceed with a discussion of the third relation and show that the values represented by the interval of convergence of the power series in it define the values for which it is a valid relation. We conclude that a *power series* can represent certain *algebraic* functions provided that we limit it to these values.

We have seen that an algebraic function can be represented by a power series and that leads us to the consideration of this question: Can functions such as e^x, $\sin x$, $\log x$, etc. be expressed as power series? Again the student may not immediately experience a conflict but when he sees that a power series is an *algebraic* expression in x and was used to represent another *algebraic* expression, he may begin to feel that perhaps a *transcendental* function, which involves operations "transcending" the laws of algebra, cannot be represented by a power series.

To answer this question we recall the definition of e^x as $\lim_{n\to\infty} (1 + x/n)^n$. Can we use a power series to express this relation?

"$(1 + x/n)^n$ can be expanded by use of the Binomial Theorem to give: $1 + x + n(n - 1)x^2/2!n^2 + \cdots$. Or, after taking the limit, $e^x = 1 + x + x^2/2! + \cdots$."

Now we have seen that functions other than those which are algebraic can be expressed as power series. Can we expand a trigonometric function as a power series? That is, is there a relation such as: $\sin x = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$?

"Maybe there is but how would we get the values of the a's?"

Is there a way that we could obtain a_0?

"Let x equal 0."

This evaluates the first constant and we then have $\sin x = a_1x + a_2x^2 + a_3x^3 + \cdots$. Now how would we get a_1?

"If the right-hand side of the relation began with a_1, we would be able to repeat the process that we used in getting a_0. If we divide by x it would seem to do it."

If we do that we will have $(\sin x)/x = a_1 + a_2x + a_3x^2 + \cdots$. Having used this to evaluate a_1, we proceed to the next one.

Now how will we obtain a_2?

"We could divide by x again."

This gives: $(\sin x)/x^2 = 1/x + a_2 + a_3x + a_4x^2 + \cdots$. Can we use this relation to evaluate a_2?

"No, because of the term $1/x'$."

It seems to me that the procedure for arriving at the expansion for $\sin x$ as outlined above is most likely to evolve although most textbooks assume that we would immediately proceed to use differentiation. In fact, I outlined this procedure before using it in the classroom and subsequently found that my classes actually did follow this line of reasoning. That is why I followed out the method of division although, as we have seen, it did lead the class up a blind alley. If we proceed this way, however, we are not forcing things to go according to a pre-conceived plan; we are letting the ideas of the class come out as they will so that, in the end, they will have a deeper understanding because they will know what it is as well as what it is not. They also get the experience of rejecting a hypothesis because it does not work.

Let us go on with the development of $\sin x$. Now, when we were trying to evaluate a_1 we made the right-hand side start with a_1 by dividing a_1x by x. Could we have applied any other process to this term that would have made it start with a_1?

"We could differentiate it."

This leads to further questions but the procedure follows smoothly from this point on and we proceed to obtain the value of a_1 and then go on to derive the complete development for $\sin x$ and also the generalized form for any Maclaurin Series.

The reader can see how the question as to whether or not we can use this method for *any* function will lead immediately into a discussion of Taylor's Series.

Assuming that this method is valuable in the teaching of mathematics, the question might well be asked: Can we apply this method to the teaching of *all* topics in mathematics?

It seems to me that every topic worthy of inclusion in the mathematics curriculum should represent a real problem for the learner - that is, it should be one which results from inadequacies and/or disharmonies in the outlook of the learner. A study of ideas in contrast or conflict seems to be very effective in revealing these inadequacies and disharmonies and also provides impetus to the study. When this study is carried forward reflectively, I contend that we are then promoting the highest type of learning under conditions most conducive to success for we are not only making teaching the student *how to think* central in our program but also creating a learning situation charged with spontaneous inquisitiveness.

It is not the contention of the writer that this idea is basically novel but merely that conflicts are inherent in every learning situation and possess great potentialities for motivation. Teachers of mathematics and writers of mathematics textbooks are urged to exploit these latent possibilities as one means of improving the instructional program.

University of Missouri

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Introduction to Number Theory. By Trygve Nagell, John Wiley & Sons, New York 1951, 309 pages, $5.00.

This textbook is indeed an excellent introduction to the subject. After an introductory chapter entitled "Divisibility", there follow chapters on the distribution of primes (essentially Tchebychef's results), congruences, quadratic residues and cyclotomy. Then comes a chapter on diophantine equations of degree two, and a chapter on diophantine equations of higher degree in two and three unknowns. This last topic is Nagell's specialty, and is very well handled. The book concludes with the Erdös-van der Corput version of Selberg's proof of the prime number theorem.

A stimulating feature is the inclusion of many quite recent results. As well as the "elementary" proof of the prime number theorem, Niven's proof of the irrationality of π is included, and Mills' prime giving formula is mentioned, though without proof. In addition, the definitive results of Siegel and Weil on lattice points on algebraic curves are described. There are several excellent collections of problems of all degrees of difficulty.

No account of continued fractions is given, nor of the systematic theory of quadratic forms. But Pells equation and Legendre's equation $ax^2 + by^2 + cz^2 = 0$ are thoroughly treated. Ample justice is done throughout the book to the contributions of Scandanavian mathematicians to the theory of numbers.

Morgan Ward

Lectures in Abstract Algebra. By Nathan Jacobson, D. van Nostrand, New York, 1951, 217 pages.

This book, the first of three projected volumes on the subject by Jacobson, is a very good general introduction to modern algebra. After the necessary (mildly Bourbakian) preliminaries, there follow four chapters on semi-groups and groups; rings, integral domains and fields; extensions of rings and fields; elementary factorization theory. Then follows an interesting chapter on groups with operators, containing among other things Shreier's refinement theorem, the Krull-Schmidt decomposition theorem, chain conditions, direct and sub-direct products. The sixth chapter on modules and ideals gives the Hilbert basis theorem and the Noether decomposition theorems for ideals of a commulative ring with ascending chain condition.

A reader familiar with the classical treatment of modern algebra by van der Waerden, will see that Jacobson has compressed nearly all the group, ring and ideal theory in van der Waerden's two volumes into less than two hundred pages. In addition, many more recent results are included, and the treatment is throughout easy and lucid. The important subject of linear algebra is omitted; it will be treated separately in a second volume.

The book closes with a chapter on lattice theory but treats fully only those parts, directly applicable to group and ring theory in line with Orë's development of the subject.

There are well-selected groups of exercises, remarkably few misprints and no errors that the reviewer could detect.

Morgan Ward

Mathematics of Investment. By Paul R. Rider and Carl H. Fischer. New York, Rinehart and Company, Inc., 1951. XI + 359 pages. $5.00.

The content of the text and the order of the topics, with few exceptions, seen to be that of the traditional text devoted exclusively to the subject. Since the text begins with simple interest and simple discount followed by compound interest and annuities, a working knowledge on the part of the student of algebra including logarithms and progressions is assumed.

Following the work on annuities, amortization and sinking funds, bonds, depreciation and capitalized cost, is extensive work in life annuities, life insurance net premiums, and life insurance reserves. There are over 1000 exercises in the text. It may be used for a two-semester course or, of course, may be adapted for a one-semester course.

Considerable care seems to have been exercised in dividing the text into normal lesson units. The development of annuities and related topics is performed in a very excellent manner from the point of view of clarity and consistency. In this connection, this reviewer particularly liked the aid of time diagrams. The cases of the general annuities have been simplified and unified.

The authors have evidently devoted a lot of time to the selection of problems. This is well illustrated by the chapter on bonds where, as the authors say, "..... the reader will not find the misuse of the word 'dividend' for the periodic interest payment on a bond, nor will the examples give the erroneous impression that the typical bond matures for an amount higher than its par value." Answers are provided only for the odd-numbered problems throughout the text.

This text has an arrangement of tables which the reviewer considers an improvement over the conventional method of interest tables by functions subdivided by interest rates. Table II, for example, is subdivided by functions and gives the values of s, v^n, $s_{\overline{n|}}$, $a_{\overline{n|}}$, and $s_{\overline{n|}}^{-1}$.

Another feature is the omission of tabular values for $a_{\overline{n|}}^{-1}$ on the supposition that first, this is an aid in giving a simple unifying treatment of the general annuity case and second, it is easier to add the interest rate to $s_{\overline{n|}}^{-1}$ to find the function $a_{\overline{n|}}^{-1}$ than it is to perform the corresponding subtraction. Too, the lowest rate shown in the table is 1% converted monthly.

W. H. Bradford

Fourier Transforms. By Ian N. Sneddon, International Series in Pure and Applied Mathematics, McGraw-Hill 1951, pp xii + 542.

In recent years, integral transform methods have superseded the older and more intuitive operational procedures for the solution of ordinary and partial differential equations in Applied Mathematics. Most books on the subject emphasize the role of the Laplace transform, implying in most cases transformation of the equation with respect to the time variable. By contrast, the book under review is concerned mainly with Fourier and Hankel transforms normally involving transformation with respect to a space variable.

Following mathematical introductions to the various types of integral transforms, including finite Fourier and Hankel transforms, there are extensive applications to the Theory of Vibrations, to Heat Conduction, to Hydrodynamics and Elasticity, and to certain problems in Atomic and Nuclear Physics including cosmic ray showers. Other stochastic problems considered are the slowing down of neutrons and the phenomenon of turbulent fluid flow. The chapter on Elasticity is largely concerned with the theory of Griffith cracks.

By discussing advanced problems in which he is personally interested from a physical point of view, the author has raised his work beyond the usual level of books on operational mathematics. On the other hand, it should be clear to the reader that the various chapters cannot always be regarded as balanced introductions to the subjects with which they are concerned (a case in point is the section on turbulence). Also, the mathematical introduction at the beginning is not in fact intended to be comprehensive and for more delicate questions reference should still be made to the book on Fourier integrals by E. C. Titchmarsh. However, it must be said that such limitations are almost inevitable in a work of this nature. In the reviewer's opinion the book represents a valuable addition to the literature and, incidentally, is a remarkable testimonial to the versatility of its author.

BRIEF SURVEYS

As a means of preventing excessive backlogs of reviews, we shall from time to time earmark the books on hand for brief surveys.

Nomography and Empirical Equations. By Lee H. Johnson, John Wiley and Sons, Inc., New York 1952, 150 + ix pages. $3.75.

For some time a revision of Joseph Lipka's book of 1918, *Graphical and Mechanical Computation*, has been in order. To some extent, the volume under review is such a revision, indeed in the preface Dean Johnson states that he has patterned this new work after certain portions of Lipka's book. The author is the Dean of the College of Engineering of Tulane University - and this new book as its "predecessor" is slanted toward engineering rather than toward mathematical aspects. The preface states that a treatment of nomography using determinants has been omitted - a fact which the reviewer regrets. Also, the methods for determining constants in empirical equations are discussed too briefly to have mathematical appeal.

Robert E. Greenwood

Statistical Calculation for Beginners. By E. G. Chambers, Cambridge University Press, 1952, x + 168 pages, $2.50.

The first edition of this book was issued in 1940, and contained viii+110 pages. It was reprinted several times before this second (and enlarged edition) was issued. A review of the first edition may be found in the *Journal of the Royal Statistical Society*, volume 104, (1941), page 64. This review closes with the sentence, "This book should be of great help to the beginner". The new material (on the binomial distribution, Kendall's rank correlation statistic, analysis of variance and elementary curve fitting of empirical data) is presented as simply as possible, according to the author's preface. As noted in the review of the first edition, the author's choice of the statistics discussed is biased in favor of statistics useful in experimental psychology.

Robert E. Greenwood

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

NUMBERS GAME*

When Shakuntala Devi was five years old in Bangalore, India, she liked to sit at the side of her uncle, who was studying mathematics at the university. As a joke, he told the little girl about cube roots. One day, when Shakuntala wanted money to buy candy, she offered to help her uncle with a problem in cube root if he would finance the candy. When he laughed, she promptly wrote down the correct answer on a sheet of paper.

From then on, Shakuntala (meaning "baby brought up by the birds") was a marked little Brahman. Her odd mathematical ability improved rapidly with practice, and soon she was giving demonstrations all over India. Later, she moved on to Europe, confounding mathematics experts and makers of computing machines.

Last week sari-clad Shakuntala, now 20, gave a demonstration in Washington before a party of reporters and mathematics professors. She had become a master of the arm-long number. Without error or hesitation she extracted fourth, fifth and sixth roots of numbers up to ten digits. (Her record to date: extracting the 20th root of a 42-digit number and multiplying figures that yielded a 39-digit result.) Without hesitation, she worked out "magic squares" (horizontal, vertical and diagonal sums are identical), starting with random numbers suggested by the audience.

Like most number prodigies, Shakuntala does not know how she does it. She thinks about the problem and the numbers come – in three or four seconds. Often she gives the answer as soon as her questioner has written down the last digit. In the case of the root problems, the answer must be a whole number. Her mysterious talent does not yield uneven answers. She has studied logarithms, but they confused her and she does not use them.

Mathematicians suggest that Shakuntala may have a fantastic memory, big enough to store all possible answers to all the tricks she offers. But this, they say, would be a startling feat in itself, probably as difficult as doing the computation in her head. Shakuntala herself, off on a cross-country tour of the U. S., just tries to avoid all discussion for fear it will disturb her strange talent. Says she: "I do not know my limits."

*Reprinted from Time Magazine, July 14, 1952.

LIQUOR IN EDUCATION

One hundred and twenty three years ago, when good pure whiskey was bought by the ton, school-children labored over problems like the following: "Having a desire to collect a number of men for a frolick, and wishing to keep them sober; I had a barrel containing 32 gallons of whiskey out of which I took 5 gallons and replaced it with 5 gallons of water; and finding it too strong, took out 5 gallons more putting in 5 gallons of water; I demand the quantities of whisky and water in the barrel."

The above problem is representative of numerous others found in "Connolly's Arithmetic" of 1829. The prevalence of such problems shows - if text books reflect the spirit of the times in which they are published - that Temperance Unions were not at all strong 100 or so years ago. It also shows that drinking to the "frolicking" stage was not considered a disgrace. The word "frolick" is no longer seen in the society columns of the newspapers; perhaps the use of the word went out with the serving of whiskey by the barrel, for even in these times there would certainly be something resembling a frolic if a barrel of whiskey were served at a party.

Examination of other Arithmetics of Connolly's time and of earlier times reveals that Connolly was not alone in demanding answers to liquor problems; In "Daboll's Schoolmaster's Assistant", published in 1820, are found about 40 of such problems, a number of which are given below: I bought 3 hogsheads of brandy containing 61, 62, and 63½ gallons at $1.38 a gallon, I demand what they amount to.
A wine merchant bought 721 pipes of wine for $90,846; and sold 543 pipes thereof for $89,649; how many pipes has he unsold and what do they stand him in?

It might be added here that it takes two hogsheads to make a pipe and two pipes or four hogsheads to make a ton; and the small amounts which they "stand the merchants in" when selling such large quantities may surprise the modern educator, but both Connolly and Daboll insist in their respective title page that their text book is "A plain, practical system."

A similar claim to practicality is made by Daniel Adams M.D. in his "Adams New Arithmetic" copyrighted in 1827. This book with its numerous problems touching on liquors is "calculated to give the pupil a full knowledge to all the practical purposes of life". Like the other mathematicians of his time, Adams offers wine measure for measuring all spiritous liquors, ale and beer excepted, and ale or beer measure for ale, beer, and milk; and does not seem at all worried about Temperance Unions.

However another "New Arithmetic" published in 1846 by Charles Davies L.L.D. does show some diminution in the number of such problems; although many of those given continue to deal with liquor in large quantities.

The following is a fair example of the kind given: Bought a hogshead of brandy at $1.25 a gallon and sold it for $78; was there a loss or gain?

Davies text book was one of the last to offer school children such problems. This of course was due to the steadily growing movement against the manufacturing and selling of spiritous liquor in the country.

Isabel C. Sturm

Dear Mr. James

I derive great pleasure and benefit from the reading of the expository articles published in the Mathematics Magazine. In connection with this matter I wish to make a suggestion which undoubtedly will interest many readers of your Magazine. In the plastic stress analysis of structures inequalities of the following type are common

$$\begin{aligned}
-M_{01} &\leq a_{11}M_1 + a_{12}M_2 + a_{13}M_3 + \cdots \leq M_{01} \\
-M_{02} &\leq a_{21}M_1 + a_{22}M_2 + a_{23}M_3 + \cdots \leq M_{02} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-M_{0n} &\leq a_{n1}M_1 + a_{n2}M_2 + a_{n3}M_3 + \cdots \leq M_{0n}
\end{aligned}$$

in which M_1, M_2, M_3 $\cdots$ M_n are the unknowns. Since textbooks on algebra do not treat this subject adequately, there is a great need for an algorithm for solving these inequalities by a successive elimination of the unknowns. The article by L. L. Dines "Systems of Linear Inequalities" Annals of Mathematics 1918-19 p. 191 is concerned with existence theorems. In addition he uses determinants which are not good in numerical computations. I hope therefore that you will publish something in your Magazine in the near future.

Thanking you very much

I am

Yours very truly

A. Floris

Correction to AN INVERSION OF THE LAMBERT TRANSFORM, "Mathematics Magazine," 1950, p. 180.

Dr. W. B. Pennington has pointed out to the author that the proof of Theorem 3.3 assumes the continuity of $a(t)$ at t_0/k, $k = 1, 2, \cdots$ rather than at t_0 alone, as stated. A neater theorem would result if $a(t)$ were assumed continuous for all positive t.

D. V. Widder

PROBLEMS AND QUESTIONS

Edited by

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction. Readers are invited to offer heuristic discussions in addition to formal solutions.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

140. *Proposed by R. B. Herrera, Los Angeles City College.*

$ABCDEF$ is a regular hexagon. The midpoints of sides AB, CD, DE, FA are joined respectively to points E, F, B, C forming a small rhombus at the center of the hexagon. Express the area of this rhombus in terms of a side of the hexagon. [This is problem 15 of the May 16, 1952 William B. Orange Mathematics Prize Competition.]

141. *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Let $abcd$, $a > 0$, be four-digit integers such that $bc < 99$ is a multiple of 9 and $a + d = 10$. Prove that for $n > 3$ all integers $(10^n - 1)abcd$ are palindromes.

142. *Proposed by George Pate, Gordon Military College, Georgia.*

A lot contains n articles. If it is known that r of the articles are defective, and that the articles are inspected in random order, one at a time, what is the probability that the k-th article $(k \geqq r)$ inspected will be the last defective one in the lot?

143. *Proposed by Leon Bankoff, Los Angeles, California.*

O is the center of the circumcircle of trapezium $ADEC$. AD and CE produced meet in B. The circumcenters of triangles ABC and DBE are O' and O'', respectively. Show that $O'O$ equals the circumradius of DBE and that $O''O$ equals the circumradius of ABC.

144. *Proposed by J. S. Cromelin, Clearing Industrial District, Chicago.*

The wind is blowing at 56 feet per second along the diagonal of a square track, around which a plane and a car are traveling in opposite directions. The plane is moving with a uniform air speed, the car with a uniform ground speed. The plane first crosses the car at the windward corner. The fourth crossing occurs at one o'clock. The seventh crossing occurs at the leeward corner, and the eighth at ten minutes after two, again at the windward corner. How large is the track?

145. *Proposed by Leo Moser, University of Alberta, Canada.*

It is well-known that $n = 14$ is the smallest even integer for which $\phi(x) = n$ is insolvable. Show for every positive integer, r, that $\phi(x) = 2(7)^r$ is insolvable.

146. *Proposed by A/3C D. L. Silverman, Lowry AFB, Colorado.*

What is the locus of the center of a circle, radius r, which touches each of three mutually perpendicular planes?

SOLUTIONS

Problem to Frustrate a Modern Archimedes

114. [Nov. 1951] *Proposed by Dewey Duncan, East Los Angeles Junior College.*

(a) In air a gold-surfaced sphere weighs 7588 grams. It is known that it may contain one or more of the metals aluminum, copper, silver or lead. When weighed successively under standard conditions in water, benzene, alcohol, and glycerine its respective weights are 6588, 6688, 6778, and 6328 grams. How much, if any, of the forenamed metals does it contain, if the specific gravities of the designated substances are taken to be as follows?

Aluminum	2.7	Alcohol	0.81
Copper	8.9	Benzene	0.90
Gold	19.3	Glycerine	1.26
Lead	11.3	Water	1.00
Silver	10.5		

(b) If the ball is given a very thin coating of plastic, to prevent amalgamating, how far will it sink when floating in mercury, specific gravity 13.6?

Solution by W. H. Quelch, Florida State University, Tallahassee. (a) Let u, v, x, y, z be the volumes in cm^3 of Al, Cu, Pb, Ag and Au, respectively, contained in the sphere, which we assume to be not hollow. Since the loss of weight in water (specific gravity 1.00) is 1000 grams, the volume of the sphere is 1000 cm^3. Then the data, some of which is superfluous though consistent, leads to only two independent equations, one relating volumes and the other, weights:

$$u + v + x + y + z = 1000$$

$$2.7u + 8.9v + 11.3x + 10.5y + 19.3z = 7588$$

Clearly the sphere must contain some aluminum to bring its mean specific gravity below the specific gravities of all the other metals. There is no unique result to this part of the problem, for the amounts of three metals may be chosen arbitrarily, provided that the choices will not result in negative amounts of any metal.

If the ball contains only aluminum and gold, there are 294.5 cm^3 of gold and 705.5 cm^3 of aluminum. Another possibility is 124.7 cm^3 each of Cu, Au, Pb and Ag and 501.2 cm^3 of Al.

(b) To find the distance, $r + h$, which the ball sinks beneath the surface, we use the equation:

$$500 + \pi \int_0^h (r^2 - x^2)dx = 7588/13.6,$$

where $r = \sqrt[3]{750/\pi} \doteq 6.203505$ cm. This leads to the cubic, $h^3 - 115.4504\, h + 55.3297 = 0$. The only root of this cubic inside $\pm r$ is $h \doteq 0.48021$. Therefore the bottom of the ball is 6.684 cm. below the surface of the mercury.

Also solved by *W. B. Carver, Cornell University; F. F. Dorsey, South Orange, N.J.; E. S. Keeping, University of Alberta; and the proposer*, who remarked that a large number of *integer* solutions are given by the parametric equations $u = 688 - 52a$, $v = 28 + 83a - 11b$, $x = 11c$, $y = 13b - 10c$, and $z = 284 - 31a - 2b - c$. The parameters are to be given integer values which yield non-negative values for u, v, x, y, z. Another single solution is $(u, v, x, y, z) = (400, 300, 200, 40, 60)$.

Peculiarity of Polyhedra

115. [Nov. 1951] *Proposed by Leo Moser, Texas Technological College.*

If all the faces of a polyhedron have central symmetry, prove that at least 6 of the faces are parallelograms. (A parallelopiped has exactly 6 such faces.)

Solution by the Proposer, University of Alberta. We use the notation and results of S. S. Cairns, "Peculiarities of polyhedra," *American Mathematical Monthly*, **58**, 684-689, (Dec. 1951), wherein V_i = the number of vertices each belonging to exactly i edges, and F_i = the number of faces each having exactly i edges (i = 3, 4, 5, $\cdots$). Thus we have:

$$V + F = E + 2 \tag{1}$$

$$V = V_3 + V_4 + V_5 + V_6 + \cdots \tag{2}$$

$$F = F_3 + F_4 + F_5 + F_6 + \cdots \tag{3}$$

$$2E = 3F_3 + 4F_4 + 5F_5 + 6F_6 + \cdots \tag{4}$$

$$2E = 3V_3 + 4V_4 + 5V_5 + 6V_6 + \cdots \tag{5}$$

Now multiply (1) by 6, and in it replace V and F by their values from (2) and (3) and $6E$ by $2E$ from (4) and $2(2E)$ from (5). This yields $3F_3 + 2F_4 + F_5 = 12 + (F_7 + 2F_8 + 3F_9 + \cdots) + 2(V_4 + 2V_5 + 3V_6 + \cdots)$.

Now a polygon having central symmetry has an even number of edges. Hence for the polyhedra in question, $F_i = 0$ for i odd. Thus we have

$$F_4 = 6 + (F_8 + V_4) + 2(F_{10} + V_5) + 3(F_{12} + V_6) + \cdots$$

so that $F_4 \geqq 6$ as required, since a quadrilateral having central symmetry is a parallelogram.

Also solved by *William Moser, Student, University of Toronto.*

A Simple Theorem Regarding the G.C.D.

116. [Nov. 1951] *Proposed by H. H. Berry, University of Kentucky.*

If a, b, and c are integers such that $a + b = c$ and the least common multiple of a and b is M, where $ab/(a,b) = M$, then $(c,M) = (a,b)$. [The symbol (a,b) represents the greatest common divisor of a and b.]

Solution by J. S. Shipman, Laboratory for Electronics, Inc., Boston, Mass. Suppose first that $(a,b) = 1$. Then $M = ab$, and $(c,M) = (c,ab)$. Now c can have no factor in common with either a or b. For if c had a factor in common with a, we could write $c = c'd$, $a = a'd$, $d > 1$, and then $b = c - a = d(c' - a')$. Thus b has a factor $d > 1$ in common with a, contrary to assumption. Similarly, $(c,b) = 1$. It follows that $(c,ab) = 1$. Therefore, $(c,M) = (c,ab) = 1 = (a,b)$.

Now suppose that $(a,b) = d > 1$, and write $a = a'd$, $b = b'd$, $(a',b') = 1$. Then $c = a + b = (a' + b')d = c'd$. Further, $M = a'b'd^2/d = a'b'd$. It follows that $(c,M) = (c'd, a'b'd) = d(c', a'b') = d$. Therefore $(c,M) = d = (a,b)$.

Also solved by *A. L. Epstein, Geophysical Research Directorate, Cambridge, Mass.; R. F. Reeves, Columbus, Ohio; and the proposer,* who remarked that this theorem is useful in solving such problems as number 9, page 41, Uspensky and Heaslet, *Elementary Number Theory.* Namely, "The sum of two numbers is 5,432 and their l.c.m. is 223,020. Find the numbers." By the above theorem, $M = ab/(5432, 223020) = 223020$ so $ab = 28(223020)$. When this equation is solved simultaneously with $a + b = 5432$, we obtain the numbers 1652 and 3780.

A Triangular Number Relationship

119. [January 1952] *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Solve $t_x t_{x+1} t_{x+2} = 468 t_y^3 + 468$ for x and y. [$t_a = a(a + 1)/2$.]

Partial solution by Leon Bankoff, Los Angeles, Calif Since $R = 468(t_y^3 + 1) = 4 \cdot 9 \cdot 13\{[y(y + 1)/2]^3 + 1\}$, inspection reveals that R

can terminate only in 4, 6, or 8. Now $L = t_x t_{x+1} t_{x+2} = x(x+1)^2(x+2)^2(x+3)/8$, so L can terminate only in 0 or 8. But $L = R$, so both terminate in 8 and $x \equiv 1 \pmod 5$. Furthermore, $L \equiv 0 \pmod{13}$, so $x \equiv 10, 11, 12$, or $0 \pmod{13}$, whereupon $x \equiv 36, 11, 51$, or $26 \pmod{65}$. We proceed to test the possible values of x and immediately find that for $x = 11$, $L = 11(12)^2(13)^2 14/8 = 468[(4 \cdot 5/2)^3 + 1] = R$, so $y = 4$. There are no other solutions for $x < 352$.

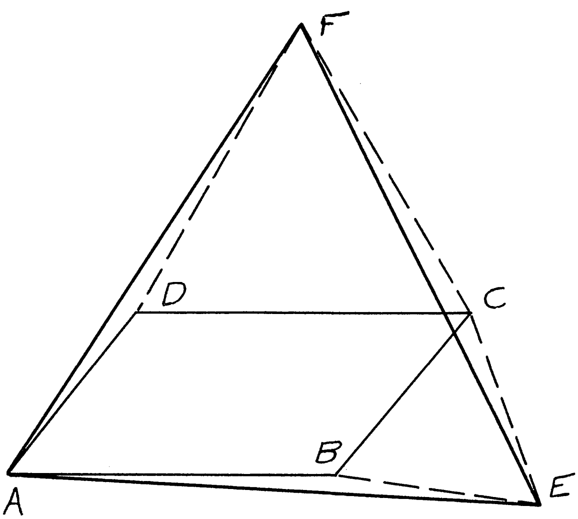
Also partially solved by *the proposer.*

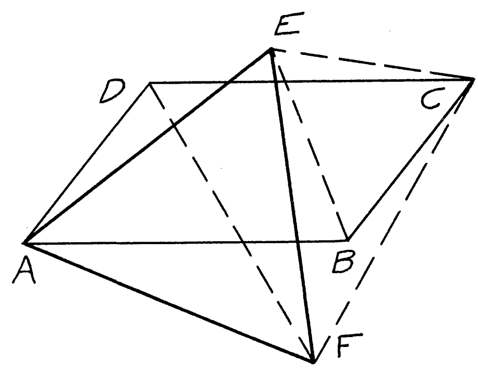
Editorial note: A general solution of this problem, or a proof that the given solution is unique will be welcome.

Equilateral Triangles Associated with a Rectangle

120. [January 1952] *Proposed by Victor Thébault, Tennie, Sarthe, France.*

On the sides CB and CD of rectangle $ABCD$ construct internally (or externally) equilateral triangles CEB and DFC. Show that triangle AEF is equilateral.



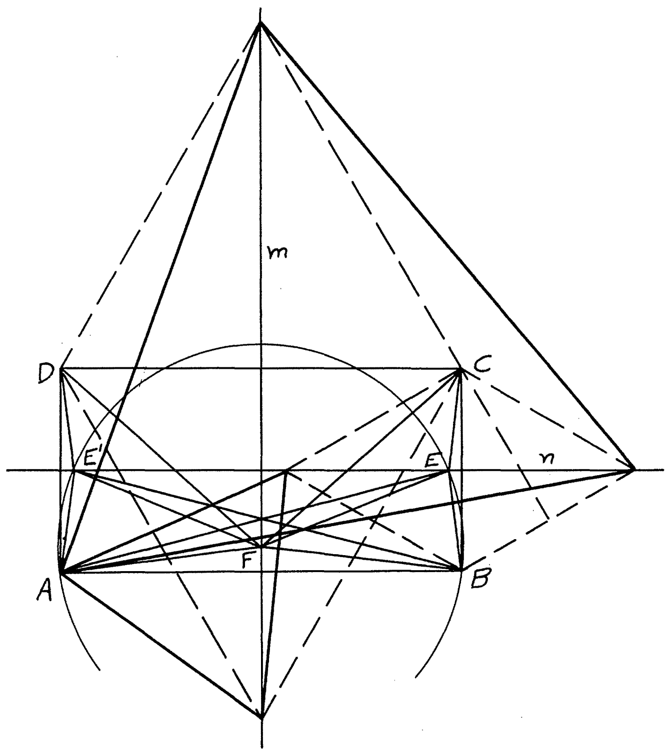


I. *Solution by G. W. Courter, Baton Rouge, Louisiana.* We generalize the proof to the parallelogram $ABCD$. In the external case, $\angle FCE = (180° - \angle ABC) + 60° + 60° = 300° - \angle ABC$, $\angle ABE = 360° - (\angle ABC + 60°) = 300° - \angle ABC$, and $\angle FDA = 360° - (\angle ADC + 60°) = 300° - \angle ABC$.

In the internal case, $\angle FCE = 60° + 60° - (180° = \angle ABC) = \angle ABC - 60°$, $\angle ABE = \angle ABC - 60°$, and $\angle FDA = \angle ADC - 60° = \angle ABC - 60°$.

Hence, in both cases, $\angle ABE = \angle FCE = \angle FDA$. Now $AB = CD = FC = FD$ and $BE = CE = BC = DA$. It follows that triangles ABE, FCE and FDA are congruent. Whereupon $AE = EF = FA$ and triangle AEF is equilateral.

The proof applies to the figures, where $\angle ABC > 120°$. When $60° < \angle ABC < 120°$, $\angle ABE = \angle FCE = \angle FDA = \angle ABC \pm 60°$; when $\angle ABC < 60°$, $\angle ABE = \angle FCE = \angle FDA = 60° \pm \angle ABC$, where the positive sign applies to the external case and the negative sign to the internal case. The remainder of the proof proceeds as before.



II. *Solution by C. D. Smith, University of Alabama.* The problem is a special case of a more general configuration. Consider m and n the respective perpendicular bisectors of the sides AB and BC of the rectangle $ABCD$. Choose F on m and describe a circle (F) with radius FA and cutting n in E and E', E nearer CB. So (F) is the circumcircle of isosceles trapezoid $ABEE'$. Then we have a system of isosceles triangles: EBC congruent to $E'AD$, AEF congruent to $BE'F$, $AE'F$ congruent to BEF, and FCD associated with the system.

Now let $\angle ABE = x$. It follows that $\angle BEC = 2x = \angle AFE$. This makes triangles EBC and AEF similar in general. Triangle FCD is isosceles but not generally similar to BEC. When $x = 30°$, triangle EBC is equilateral (and hence triangle AEF is also) so $CE = CB$. Consequently, CF is the perpendicular bisector of BE, which makes $\angle FCD = 60°$ and triangle FCD equilateral. This proves the interior case.

When E moves along n until it falls outside the rectangle and $x = 150°$, then triangles EBC and AEF are equilateral. Again CF is the perpendicular bisector of BE, so $\angle FCD = 60°$ and triangle FCD is equilateral. This proves the exterior case.

The general configuration is a problem of three circles on point E and determined by the vertices A, B; B, C; and A, D of the rectangle. For a discussion of three-circle problems with a triangle of reference, see *National Mathematics Magazine*, **14**, 299-307, (March 1940).

Also solved by *Leon Bankoff, Los Angeles, Calif.; F. F. Dorsey, South Orange, N.J.; A. L. Epstein, Cambridge Research Center, Mass.; Mary Hanania, Beirut College for Women, Lebanon; R. E. Jackson, Student, University of California at Los Angeles; Samir Khabbaz, Bethel College, Kansas; M. S. Klamkin, Polytechnic Institute of Brooklyn; R. R. Phelps, Student, Los Angeles City College; Charles Salkind, Polytechnic Institute of Brooklyn; Perry Seagle and N. D. Stanton, Students, Winthrop College, S.C.; R. P. Smith, Student, Drake University, Iowa; and the proposer.*

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

"**Q 59.** [March 1952] Across one corner of a rectangular room two 4-foot screens are placed in such a manner as to enclose the maximum floor space. Determine their positions." *Charles Salkind* offers this alternative solution. The two screens and the two walls determine an isosceles triangle and a right triangle, the hypotenuse of which is the base of the isosceles triangle. The maximum right triangle with a given hypotenuse is isosceles. Thus we have two isosceles triangles with vertices on opposite sides of their common base. The second diagonal of the quadrangle thus formed bisects the right angle. Now we have two triangles each with a fixed 4-foot base and a vertex angle of 45°. To be a maximum each of these triangles must be isosceles with remaining angles of 67½°, these being the angles made with the walls by the screens.

Q 67. Find a solution in positive integers of $a^3 + b^4 = c^5$. [*Submitted by Leo Moser.*]

Q 68. A courier rode from the rear of a column of marching soldiers to the front and returned forthwith to the rear of the column. He kept his horse jogging along exactly three times as fast as the column itself was advancing. Where on the road, with reference to the original position of the vanguard, did he complete his journey? [*Monte Dernham in THE BAT, No. 63, page 472, March 1949.*]

Q 69. The area of the surface bounding the space common to two circular cylinders of unit radii whose axes meet at right angles is 16. [*Submitted by J. H. Butchart.*]

Q 70. Find the maximum value of $\left(\sum_{n=1}^{N} a_n x_n\right) \prod_{n=1}^{N} (a_n - x_n)$. [*Submitted by M. S. Klamkin.*]

ANSWERS

A 67. Since $2^{24} + 2^{24} = 2^{25}$, we have $(2^8)^3 + (2^6)^4 = (2^5)^5$ or $a = 256$, $b = 64$, $c = 32$.

A 68. The column traveled ½ of its length while the courier traveled 3/2 its length to reach the front. Then the column advanced 1/4 its length while the courier traveled 3/4 its length to reach the rear. Hence the column advanced 3/4 its length while the courier traveled and completed his journey 1/4 length to the rear of the vanguard's original position.

A 69. The common space can be subdivided into infinitesimal pyramids with vertices at the intersection of the axes and bases along the elements of the cylinders. These pyramids have unit altitude. Now the volume of the space is 16/3 [see **Q 66**, page 290, May-June 1952], hence the area is 16.

A 70. $\left(\sum_{n=1}^{N} a_n x_n\right) \prod_{n=1}^{N} (a_n - x_n) = \left(\sum_{n=1}^{N} a_n x_n\right) \prod_{n=1}^{N} (a_n^2 - a_n x_n) / \prod_{n=1}^{N} q_n$.

Clearly the sum of the factors of the numerator is a constant, $\sum_{n=1}^{N} a_n^2$. Now it follows from the theorem that the geometric mean $\leq$ the arithmetic mean, that if the sum of k factors of a function is a constant, b, then the maximum value of the function is $(b/k)^k$. Therefore the maximum value sought is $\left(\sum_{n=1}^{N} a_n^2\right)^{N+1} / (N+1)^{n+1} \prod_{n=1}^{N} a$.